Lesson 19 Complex power series 5.1

15, 19, 20 due next Wed. (no 16)

Read 5.1 and 5.2 carefully

Sequence $\{a_n\}_{n=1}^{\infty}$. $\lim_{n\to\infty} a_n = L \in \mathbb{C}$ means: Given $\varepsilon > 0$, there is an N such that $|a_n - L| < \varepsilon$ when $n > N$.

<u>Fact</u>: All basic properties of real limits are true for complex limits.

<u>EX</u>: $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$

1) $\lim_{n\to\infty} a_n b_n = A \cdot B$ 2) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided $B \neq 0$

<u>Power Series</u>: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

<u>EX</u>: Geometric series: $1 + z + z^2 + \cdots + z^N = S_N$

$$1 + z + \cdots + z^N = S_N$$
$$z + \cdots + z^N + z^{N+1} = z S_N$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$1 - z^{N+1} = (1 - z) S_N$$

$$S_N = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{\varepsilon_N(z)}$$

$$\left| S_N - \frac{1}{1-z} \right| = \left| \frac{z^{N+1}}{1-z} \right|$$

<u>Denominator estimate</u>: $|z - w| \geq \big| |z| - |w| \big|$

$$|1 - z| \geq \big| 1 - |z| \big| = 1 - |z| \text{ if } |z| < 1.$$

So $$|\varepsilon_N(z)| = \frac{|z|^{N+1}}{|1-z|} \leq \frac{|z|^{N+1}}{1 - |z|} \xrightarrow[N\to\infty]{} 0 \text{ if } |z| < 1.$$

Fact: $\sum_{n=0}^{\infty} z^n$ converges uniformly to $\frac{1}{1-z}$ on $D_r(0)$ with $0<r<1$.

Why: $|\mathcal{E}_N(z)| \le \frac{|z|^{N+1}}{1-|z|} < \frac{|z|^{N+1}}{1-r} < \frac{r^{N+1}}{1-r} \to 0$ as $N\to\infty$.

if $|z|<r$ if $|z|<r$

Fact: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n\to\infty$.

One way implication! $\Longrightarrow$ true. $\Longleftarrow$ not true: $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$

Why: $\sum_{n=1}^{\infty} a_n = L$ $S_N = \sum_{n=1}^{N} a_n$.

$$S_N - S_{N-1} = a_N$$

$$0 = L - L \quad \text{as } N\to\infty.$$

Consequence: Geometric series diverges if $|z|\ge 1$ because $|z^n| = |z|^n \ge 1$ if $|z|\ge 1$. Terms don't go to zero.

There is a "radius of convergence" $R=1$

Remark: $\sum_{0}^{\infty} z^n$ does not converge uniformly on $D_1(0)$

Why: $|\mathcal{E}_N(z)| = \dfrac{|z|^{N+1}}{|1-z|}$ ← boom! $z=1$

Def^n: A sequence of functions $f_n(z)$ on a domain Ω converges uniformly on compact subsets to $f(z)$ if:

for each compact $K \subset \Omega$ [Compact = closed & bounded]
and $\varepsilon > 0$, there is an N such that

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for } n > N$$

for all $z \in K$.
uniformly on K

Notation: $f_n \twoheadrightarrow f$

Remark: $\Omega = D_1(0)$. Unif conv on compacts $\Longleftrightarrow$ unif conv on $D_r(0)$ for each $r<1$.

Fantastic complex variable fact: Morera's Theorem $\Longrightarrow$

If a sequence $f_n(z)$ of analytic fcns on a domain Ω converge uniformly on compact subsets to f, then f is analytic.

Why: Restrict attention to a disc $\overline{D_R(a)} \subset \Omega$.

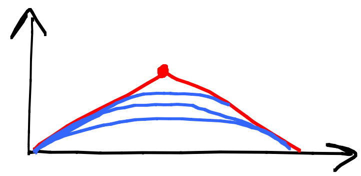
Important fact: Uniform limit of continuous fcns is continuous.

Suppose γ is a closed curve in $D_R(a)$. $tr(\gamma)$ is compact!

$$\left| \int_\gamma f\,dz - \underbrace{\int_\gamma f_n\,dz}_{=0 \text{ by Cauchy Thm}} \right| = \left| \int_\gamma (f - f_n)\,dz \right| \le \underbrace{\left(\max_{tr(\gamma)} |f - f_n|\right)}_{\text{get unif error est on compact set}} \text{Length}(\gamma)$$

Conclude that $\int_\gamma f\,dz = 0$. Morera's $\Rightarrow$ f analytic!

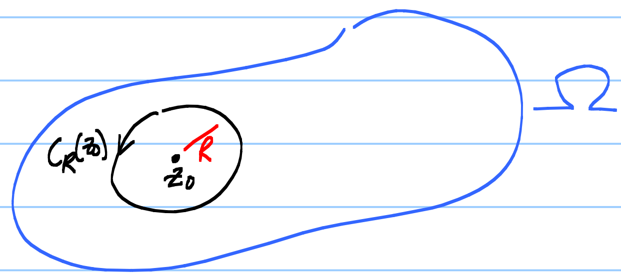
Way false $\mathbb{R} \to \mathbb{R}$:



blue are C^∞ smooth
red not diff'ble!

Possible to cook up C^∞ fcns that conv uniformly to a nowhere diff'ble fcn. Weierstraß: $\sum \frac{1}{2^n} \sin(2^n x)$

Huge fact:



$\overline{D_R(z_0)} \subset \Omega$

If f is analytic on Ω, then f is given by a convergent power series on $D_R(z_0)$:

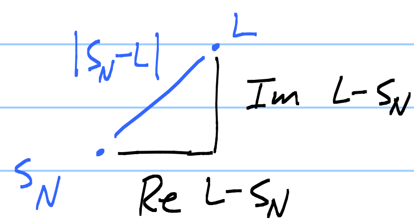
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

with unif conv on $\overline{D_R(z_0)}$.

Why: Cauchy Integral Formula, geometric series!

Fact: $\underbrace{\text{Absolutely convergent}}_{\sum_{n=1}^{\infty} |a_n| < \infty}$ series $\underbrace{\text{converge}}_{\sum_{n=1}^{\infty} a_n = L}$.

Why: $S_N \to L$



$$S_N \to L \iff \begin{cases} \operatorname{Re} S_N \to \operatorname{Re} L \\ \operatorname{Im} S_N \to \operatorname{Im} L \end{cases}$$

(Real version of abs conv $\Rightarrow$ conv) $\Rightarrow$ complex version.