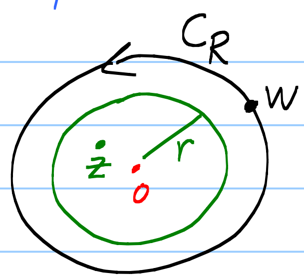
Lesson 20 Analytic fcns have power series expansions!

Assume f analytic on a disc bigger than $D_R(0)$.



$0 < r < R$

Aha! $\left|\frac{z}{w}\right| = \frac{|z|}{R} < \frac{r}{R} < 1$

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\left(\frac{1}{1-\frac{z}{w}}\right)dw$$

Notation: $$\frac{1}{1-\frac{z}{w}} = 1 + \left(\frac{z}{w}\right) + \cdots + \left(\frac{z}{w}\right)^N + \underbrace{\frac{\left(\frac{z}{w}\right)^{N+1}}{1-\frac{z}{w}}}_{\mathcal{E}_N\left(\frac{z}{w}\right)}$$

$$f(z) = \sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w^{n+1}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}} z^n + \underbrace{\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\,\mathcal{E}_N\left(\frac{z}{w}\right)dw}_{E_N(z)}$$

Finally, $$|E_N(z)| \le \frac{1}{2\pi}\underbrace{\left(\operatorname{Max}_{C_R} |f|\right)}_{M}\frac{1}{R}\underbrace{\left(\operatorname{Max}_{|w|=R} \mathcal{E}_N\left(\frac{z}{w}\right)\right)}_{\le \frac{\left(\frac{r}{R}\right)^{N+1}}{1-\frac{r}{R}}}\underbrace{\text{Length}(C_R)}_{2\pi R}$$

$$\le \frac{M\left(\frac{r}{R}\right)^{N+1}}{1-\frac{r}{R}} \longrightarrow 0 \text{ as } N\to\infty.$$

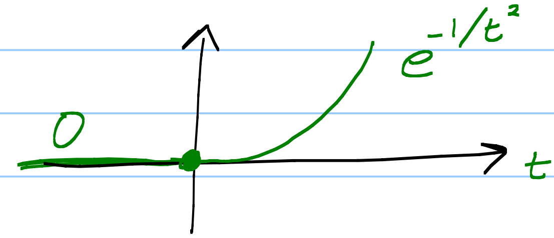
This shows that $\sum_{n=0}^{N} a_n z^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$ converges uniformly to $f(z)$ on $D_r(0)$ when $0 < r < R$.

EX: $f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

converges unif on $D_r(0)$ for any $r > 0$.

Say "Radius of Conv $= \infty$".

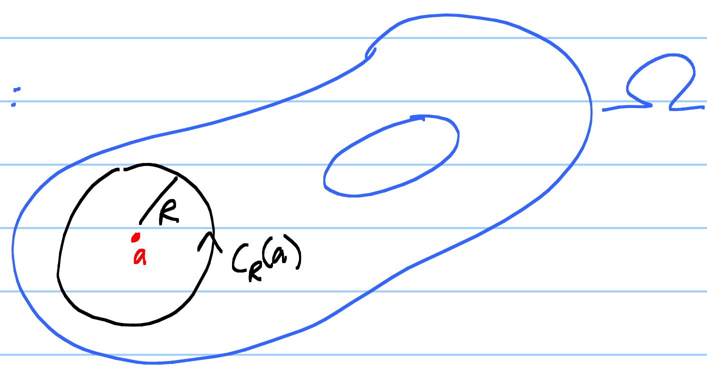
Way false $\mathbb{R} \to \mathbb{R}$:



defines an infinitely diff'ble fcn.

Power series = zero power series at $z=0$ not equal to $h(t)$.

Theorem:



f analytic on Ω

$\overline{D_R(a)} \subset \Omega$

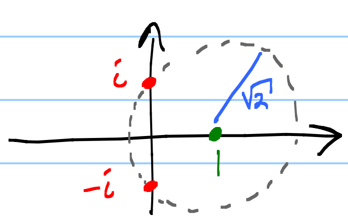
Then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ converges unif on

$D_r(a)$ with $0 < r < R$. (Radius of conv $\geq R$).

Corollary: Radius of conv of the power series about a is

$\geq \operatorname{dist}(a, b\Omega)$.

EX: $f(z) = \frac{1}{z^2+1}$ $\quad a = 1$. What is the radius of conv of the Taylor series about $a = 1$.



$\Omega = \mathbb{C} - \{\pm i\}$

$R \geq \sqrt{2}$ $\quad$ But $f(z)$ blows up at $\pm i$, so $R \leq \sqrt{2}$ too.

$R = \sqrt{2}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

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$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$