Lesson 21 Power Series

Lessons 21, 22 due Wed

$\mathbb{C}$ is a complete field: Every polynomial has a root in $\mathbb{C}$.

$\mathbb{C}$ is a complete metric space: Every Cauchy sequence converges to a limit

Def^n: A sequence $a_n \in \mathbb{C}$ is Cauchy means: for any $\varepsilon > 0$, there is an N such that $|a_n - a_m| < \varepsilon$ if n and m are $> N$.

Note: Rational $\mathbb{Q} \subset \mathbb{R}$ is not a complete field and not complete as a metric space. $\mathbb{R}$ is not a complete field, is complete metric space.

Fact: $\mathbb{R}$ complete $\Longrightarrow$ $\mathbb{C}$ complete as a metric space.

Comparison Test: $M_j > 0$ and $\sum_{j=1}^{\infty} M_j < \infty$.

If $c_j \in \mathbb{C}$ and $|c_j| \le M_j$, then $\sum_{j=1}^{\infty} c_j$ converges.

Why: $S_N = \sum_{j=1}^{N} c_j$.

$$\underbrace{|S_N - S_M|}_{} = \left| \sum_{j=N}^{M} c_j \right| \le \sum_{N}^{M} |c_j| \le \sum_{N}^{M} M_j$$

Tail end of a conv. series

S_N is a Cauchy seq. So it converges.

Ratio Test: Given a series $\sum_{n=0}^{\infty} c_n$, suppose that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L \text{ exists.}$$

If $L < 1$, the series converges.
If $L > 1$, the series diverges.
If $L = 1$, the test fails.

Pf: If $L < 1$, compare tail end of series to a convergent geometric series.

If $L > 1$, the terms don't $\to 0$, so series can't converge.

$L = 1$ cases: $\sum \frac{1}{n}$ diverges. $\sum \frac{1}{n^2}$ converges.

Hated fact: If $\lim\limits_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists and is equal to R,

then $\sum\limits_{n=0}^{\infty} a_n z^n$ has radius of convergence $= R$.

EX: For what z does $\sum\limits_{n=0}^{\infty} \underbrace{n^2 3^n z^{2n}}_{c_n}$ converge?

Hated fact is useless here because odd terms $= 0$.

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{(n+1)^2 \, 3^{n+1} \, z^{2(n+1)}}{n^2 \, 3^n \, z^{2n}} \right| = \left(\frac{n+1}{n} \right)^2 3 \, |z|^2$$

$$\xrightarrow[n\to\infty]{\text{as}} 1^2 \cdot 3 \, |z|^2 \quad \begin{matrix} < 1 & \text{conv} \\ > 1 & \text{div} \end{matrix}$$

Conv $|z|^2 < \frac{1}{3}$: $|z| < \frac{1}{\sqrt{3}} \leftarrow R = \frac{1}{\sqrt{3}}$

Div $|z|^2 > \frac{1}{3}$ $|z| > \frac{1}{\sqrt{3}}$

EX: Where does $\sum\limits_{k=0}^{\infty} z^{k!}$ converge?

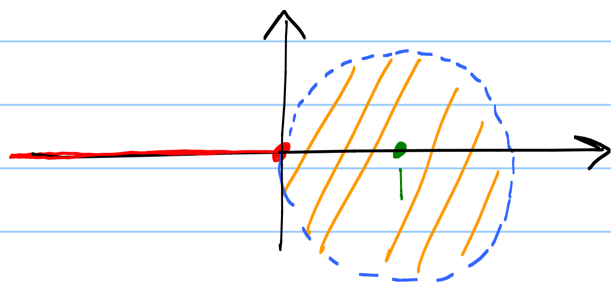
EX: $e^z = \sum\limits_{n=0}^{\infty} \underbrace{\frac{z^n}{n!}}_{c_n}$

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{z^{n+1}/(n+1)!}{z^n / n!} \right| = \frac{|z|}{n+1} \xrightarrow[n\to\infty]{\text{as}} 0 < 1 \text{ conv. } \checkmark$$

Converges for all z. $R = \infty$.

EX: $f(z) = \text{Log } z = \sum_{n=0}^{\infty} a_n (z-1)^n$ $\qquad a_n = \dfrac{f^{(n)}(1)}{n!}$

Find the series and its radius of conv.



$f(z)$ analytic on $D_1(1)$. So the Theorem from last time shows that $R \geq 1$.

But $f(z)$ blows up at $z=0$. So $R \leq 1$ too, and conclude $R = 1$.

$$f(z) = \text{Log } z \qquad\qquad f(1) = \text{Log } 1 = \text{Ln}|1| + i\,0 = 0$$

$$f'(z) = \frac{1}{z} \qquad\qquad f'(1) = 1$$

$$f''(z) = -\frac{1}{z^2} \qquad\qquad f''(1) = -1$$

$$f'''(z) = \frac{2}{z^3} \qquad\qquad f'''(1) = 2$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{z^n} \qquad\qquad f^{(n)}(1) = \frac{(-1)^{n+1}(n-1)!}{1^n}$$

So $\quad a_n = \dfrac{f^{(n)}(1)}{n!} = \dfrac{(-1)^{n+1}}{n}$

$$\text{Log } z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots$$

Famous formula for R: Hadamard's Formula $\quad \sum_{n=0}^{\infty} a_n z^n$

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

EX: Sup = "Supremum" = Least upper bound

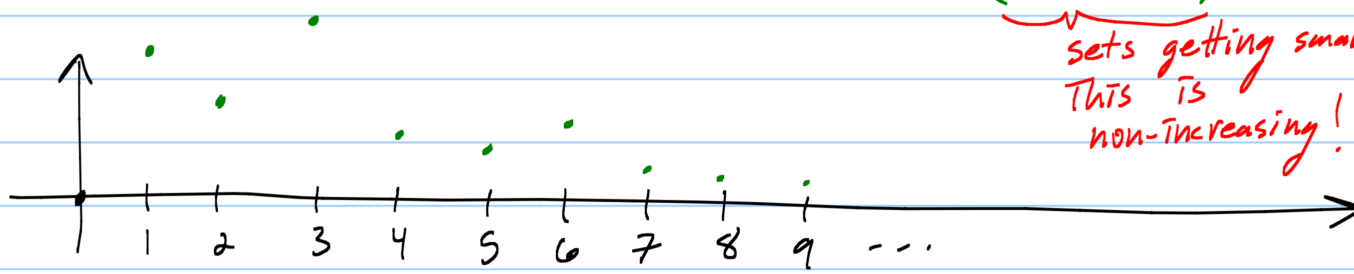
$S' = \{$ rational numbers r such that $r^2 < 2\}$ ← $\sqrt{2}$ not in set

$\sqrt{2}$ = Least upper bound. Note: $\sqrt{2}$ not in set.

$$\sqrt{2} = \text{Sup}\{r : r \in S'\}.$$

Def^n: r_n real numbers. $\displaystyle \lim_{n\to\infty}\text{Sup}\, r_n = \lim_{N\to\infty} \underbrace{\Big(\sup_{n \geq N} r_n\Big)}_{\text{sets getting smaller}}$

sets getting smaller
This is non-increasing!



Fact: A non-increasing sequence that is bounded from below has a limit.

Proof: Next time. Read 5.3 and 5.4 carefully.

EX: $\displaystyle\sum_{z=0}^{\infty} z^{k!} = 1 + z + z^2 + z^6 + z^{24} + \cdots$

$a_n = 1$ if $n = k!$ for some k
$\quad\;\; 0$ otherwise.

$\displaystyle\sup_{n > N} \sqrt[n]{|a_n|} = 1.$ $\displaystyle\lim_{n\to\infty}\text{Sup} \sqrt[n]{|a_n|} = 1.$ So $R = 1$.

Hated fact fails. Ratio test fails.

Big fact: Can differentiate power series term by term.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad R_f > 0.$$

Then $f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$ $R_{f'} = R_f$

Why: $a_n = \dfrac{f^{(n)}(a)}{n!}$ implies fact (plus Cauchy integral / Power Series fact.)