Lesson 23 Zeroes of analytic functions, isolated singularities

21, 22 due Wed
Exam 2 in class Wed., Nov. 8

Feeling: Analytic fcns given by convergent power series. So they act like polynomials!

Zeroes of analytic fcns: f analytic on $D_R(z_0)$ and $f(z_0) = 0$.

We know $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ with R. of C. $\geq R$.

Case 1: All a_n's $= 0$. Then $f(z) \equiv 0$ on R.

Case 2: Not all $a_n = 0$. Then there is a smallest N such that

$a_N \neq 0$ and $a_n = 0$ for $n = \underline{0}, 1, 2, \dots, N-1$.

$\uparrow a_0 = f(z_0) = 0$

$$\text{Then } f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$
$$= (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z)} \Big]$$

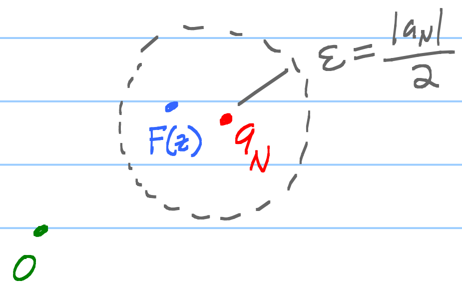
Aha! Power series for $F(z)$ converges for exactly $\overset{\text{same}}{\vee}$ z as the one for $f(z)$!

So R. of C. for series for $F(z)$ is $\geq R$ too.

Note: $F(z_0) = a_N \neq 0$.

Big conclusion: If $f \not\equiv 0$, then the zeroes of f are isolated.

Picture: $f(z) = (z-z_0)^N F(z)$ where F analytic on $D_R(z_0)$ and $F(z_0) = a_N \neq 0$.



F analytic $\Rightarrow$ continuous. So there is a δ with $0 < \delta < R$ such that

$|F(z) - a_N| < \varepsilon$ if $z \in D_\delta(z_0)$.

$\uparrow F(z_0)$

Then $F(z) \neq 0$ on $D_\delta(z_0)$

We have shown: There is a $\delta > 0$ such that z_0 is the only zero of f on $D_\delta(z_0)$. (This is the def^n of being isolated.)

Def^n: N is the order of the zero of f at z_0, or the multiplicity.

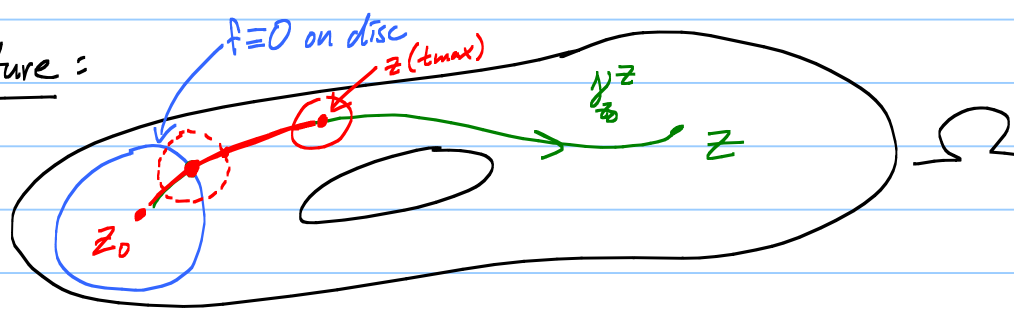
N = smallest N such that $f^{(N)}(z_0) \neq 0$ and $f^{(n)}(z_0) = 0$ for $n < N$.

Cor: If f analytic on $D_R(z_0)$ and zeroes "pile up" at z_0, then $f \equiv 0$ on $D_R(z_0)$.

Theorem: (Uniqueness Theorem or Identity Theorem) Suppose f is analytic on a domain Ω. Let $Z = \{z \in \Omega : f(z) = 0\}$. If there is a point $z_0 \in \Omega$ where the zeroes of f accumulate, then $f \equiv 0$ on Ω.

Def^n: $z_0 \in \Omega$ is called an accumulation point (or limit point) of Z if there is seq $z_n \in Z$ with $z_n \neq z_0$ for all n and $\lim_{n \to \infty} z_n = z_0$.

Pf by picture:



Suppose z_0 is a limit point of Z in Ω.

$\gamma_{z_0}^{z}$: $z(t)$, $a \le t \le b$. $f(z(t)) \equiv 0$ near a.

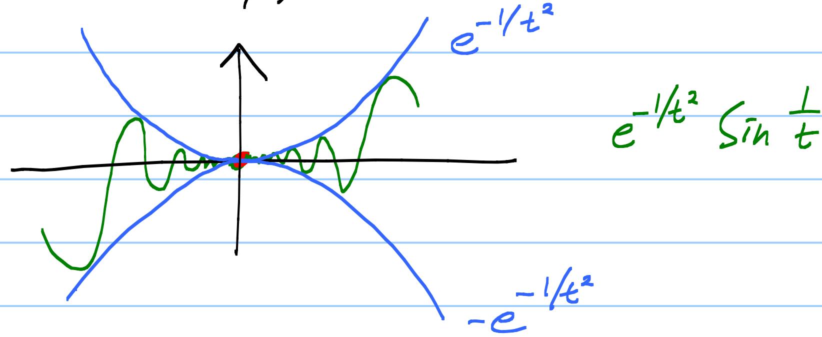
Let $t_{max} = \sup\{\tilde{\tau} : f(z(t)) = 0$ for $a \le t < \tilde{\tau}\}$.

But $z(t_{max})$ is a non-isolated zero of f. So $f \equiv 0$ near there.

See t_{max} is not the supremum! ↯. Conclude $t_{max} = b$.

So $f(z) = 0$. Since z arbitrary, see $f \equiv 0$ on Ω.

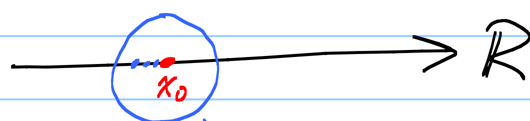
Way false $\mathbb{R} \to \mathbb{R}$:



Green fcn is C^∞-smooth and zeroes pile up at 0. Not $\equiv 0$!

Great fact: e^z is only analytic way to extend e^x from $\mathbb{R}$ to $\mathbb{C}$.

Why: If $E(z)$ extends e^x too, then $E(z) - e^z$ is analytic and vanishes on $\mathbb{R}$.



Identity Thm $\Rightarrow$ $E(z) - e^z \equiv 0$ on disc.

Cor: If f is analytic on Ω and $f \equiv 0$ on $D_\varepsilon(a) \subset \Omega$, then $f \equiv 0$ on Ω.

Great fact: All trig identities hold for complex angles.

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$\sin 2z - 2\sin z \cos z$ is analytic and vanishes $\mathbb{R}$. (Every point of $\mathbb{R}$ is a limit point of $\mathbb{R}$.) So it $\equiv 0$ on $\mathbb{C}$. ✓

$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha$ Im $e^{i\alpha} e^{i\beta}$

1. let $\alpha = z$. True on $\mathbb{C}$ for $\alpha = z$.
2. Fix z. Let $\beta = w$. True for all $w \in \mathbb{C}$.

Thm: If f and g are analytic on a domain Ω and f and g

agree on a set with a limit pt in Ω, then $f \equiv g$ on Ω.

Pf: Identity Thm using $f-g$.

Def^n: z_0 is called an isolated singularity of f if there is an $\varepsilon > 0$ such that f is analytic on $D_\varepsilon(z_0) - \{z_0\}$.

Big fact: Only 3 kinds of isolated singularities:

1) Removable: Ex: $\frac{\sin z}{z}$ at $z=0$.
2) Pole: Ex: $\frac{1}{z^3}$ at $z=0$
3) Essential Singularity: Ex: $e^{1/z}$ at $z=0$.