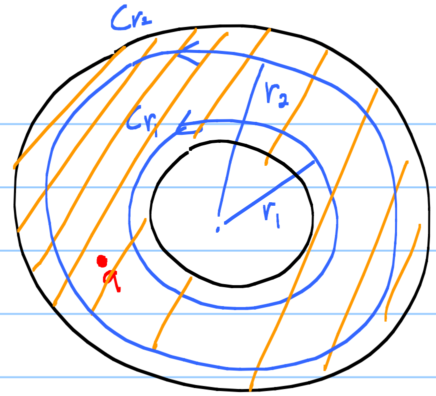
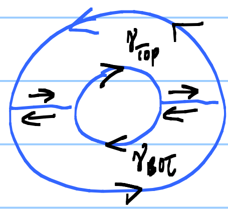
Lesson 24 Laurent expansions

23, 24, 25 due Wed



f analytic on annulus

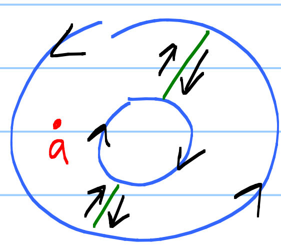
Cauchy Theorem: $\left(\int_{C_{r_2}} - \int_{C_{r_1}} \right) f \, dz = 0$



$$\underbrace{\int_{\gamma_{TOP}}}_{=0} + \underbrace{\int_{\gamma_{BOT}}}_{=0} = \int_{C_{r_2}} + \int_{-C_{r_1}} \quad \checkmark$$

Cauchy Integral Theorem:

$$f(a) = \frac{1}{2\pi i} \left(\int_{C_{r_2}} - \int_{C_{r_1}} \right) \frac{f(z)}{z-a} \, dz$$

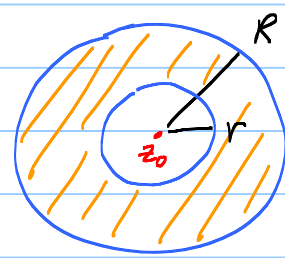


Add up 2 halves:

$$\underbrace{\frac{1}{2\pi i} \int_{\text{Left}} \frac{f(z)}{z-a} \, dz}_{f(a)} + \underbrace{\frac{1}{2\pi i} \int_{\text{RIGHT}} \frac{f(z)}{z-a} \, dz}_{=0 \text{ by Cauchy Thm}}$$

$$= \frac{1}{2\pi i} \left(\int_{C_{r_2}} - \int_{C_{r_1}} \right) \frac{f(z)}{z-a} \, dz \quad \checkmark$$

Laurent expansions:



f analytic on annulus centered at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

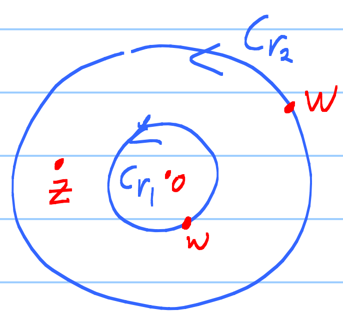
where $\displaystyle a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ for $n \in \mathbb{Z}$

where $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges uniformly on $D_\rho(z_0)$ for ρ with $0<\rho<R$ and

$\displaystyle\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{z : |z-z_0| > \rho\}$ for ρ with $r<\rho<\infty$. Note: they both converge on the annulus (overlap).

Write $\displaystyle f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

Why it's true:



$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(w)}{w-z}\,dw}_{\text{power series half}} - \underbrace{\frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(w)}{w-z}\,dw}$$

$$\frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(w)}{w}\underbrace{\frac{1}{1-\frac{z}{w}}}_{1+\frac{z}{w}+\frac{z^2}{w^2}+\cdots}dw$$

$$= \sum_{n=0}^{\infty}\Bigg(\underbrace{\frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(w)}{w^{n+1}}\,dw}_{a_n}\Bigg) z^n$$

$$-\frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(w)}{-z}\,\frac{1}{1-\frac{w}{z}}\,dw$$

$\left|\frac{w}{z}\right|<1$ on C_{r_1}

$$\frac{1}{2\pi i}\,\frac{1}{z}\int_{C_{r_1}} f(w)\left[1+\frac{w}{z}+\frac{w^2}{z^2}+\cdots\right]dw$$

$$= \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{C_{r_1}} f(w)\,w^{n-1}\,dw\right)} z^{-n}$$

$$a_{-n} = \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(w)}{w^{(-n)+1}}\,dw \quad ✓$$

Can use geom series error estimates to show unif conv. facts.

Note $\int_{C_{r_2}} = \int_{C_r} = \int_{C_{r_1}}$ since $\frac{f(w)}{w^{n+1}}$ is analytic on annulus (by Cauchy Thm).

Exciting fact: If analytic on $D_R(z_0) - \{z_0\}$, then Laurent expansion is valid on the whole punctured disc.

Three possibilities: 1) The series has no negative terms. Then f = power series. So f is analytic on $D_R(z_0)$ if we set $f(z_0) = a_0$. z_0 is called a removable singularity.

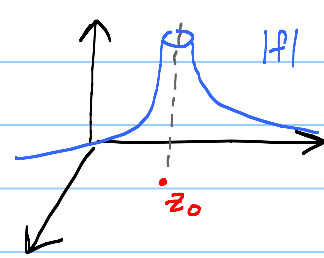
2) There are only finitely many negative terms:

$$f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)}}_{\text{rational}} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\substack{\text{regular power series} \\ g(z)}}$$

$$= \frac{1}{(z-z_0)^N}\left[\underbrace{a_{-N} + a_{-N+1}(z-z_0) + \cdots a_{-1}(z-z_0)^{N-1} + (z-z_0)^N g(z)}_{\substack{F(z) \text{ is analytic on } D_R(z_0) \\ \text{and } F(z_0) = a_{-N} \neq 0.}}\right]$$

$$= \frac{F(z)}{(z-z_0)^N} \leftarrow \text{see that } \lim_{z \to z_0} f(z) = \infty$$

We say f has a pole at z_0.



3) There are infinitely many negative terms. We say f has an essential singularity at z_0. f goes haywire near z_0!

EX: 1) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = \underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}_{\text{Power series!}}$$

Converges for all $z \neq 0$. So $R = \infty$. Entire.

Aha! Define $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$

is entire. $z = 0$ is removable sing for $\frac{\sin z}{z}$.

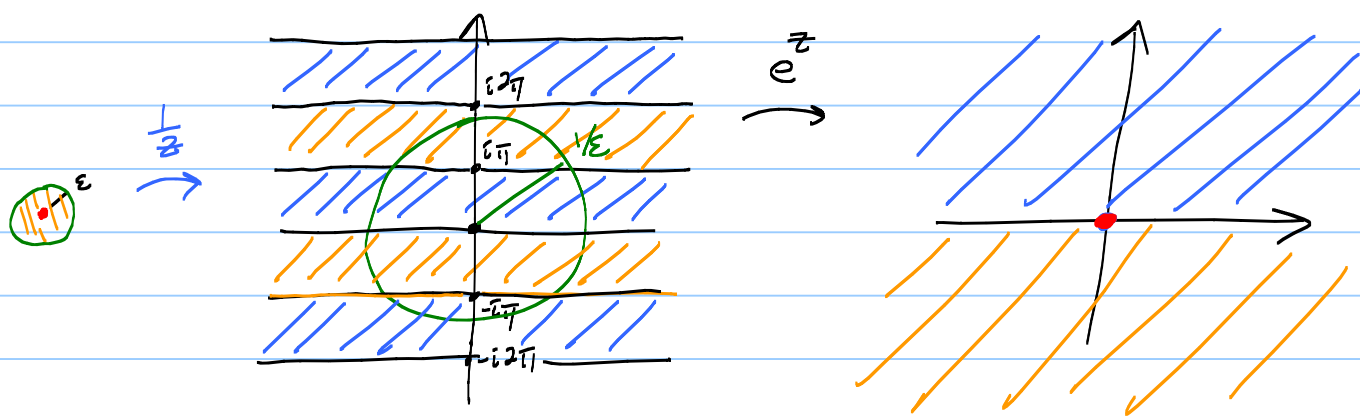
2) $$\frac{\cos z}{z^3} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3} = \underbrace{\frac{1}{z^3} - \frac{1/2!}{z}}_{\text{Principal part}} + \underbrace{\frac{1}{4!} z - \frac{1}{6!} z^2 + \cdots}_{\text{regular power series}}$$

$\frac{\cos z}{z^3}$ has pole of order 3 at $z = 0$

3) $$e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

Infinitely many negative terms in L. Exp.

$e^{1/z}$ has an essential sing at $z = 0$.



Wow! $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ onto $\mathbb{C} - \{0\}$ and is infinite-to-one!