Lesson 26 Residue integrals (6.1) 23,24,25 due Wed

f analytic on $D_r(z_0) - \{z_0\}$ ← Isolated Singularity at z_0

1) z_0 removable $\iff \lim_{z \to z_0} f(z)$ exists $\iff$ $f(z)$ bounded on $D_\varepsilon(z_0) - \{z_0\}$

2) z_0 pole $\iff \lim_{z \to z_0} f(z) = \infty$

3) z_0 essential $\iff f : D_\varepsilon(z_0) - \{z_0\} \longrightarrow \mathbb{C}$ or $\mathbb{C} - \{\text{onepoint}\}$ $\iff$ $f(D_\varepsilon(z_0) - \{z_0\})$ dense in $\mathbb{C}$ for each ε, $0<\varepsilon<r$

∞-to-one! for $0<\varepsilon<r$

Only these 3 possible behaviors

Zeroes of order N

$$f(z_0) = 0, \ldots, f^{(N-1)}(z_0) = 0$$
$$f^{(N)}(z_0) \neq 0$$

$$f(z) = a_N (z-z_0)^N + \cdots$$
$$= (z-z_0)^N \underbrace{[a_N + a_{N+1}(z-z_0) + \cdots]}_{F(z) \text{ analytic}}$$

$$\boxed{f(z) = (z-z_0)^N F(z), \quad F(z_0) \neq 0}$$

Poles of order N

$$f(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad A_N \neq 0$$

$$= \frac{1}{(z-z_0)^N} \underbrace{\left[A_N + \cdots + A_1 (z-z_0)^{N-1} + \text{analytic}\right]}_{F(z) \text{ analytic}}$$

$F(z_0) = A_N \neq 0$

$$\boxed{f(z) = (z-z_0)^{-N} F(z), \quad F(z_0) \neq 0}$$

Application of the Residue Theorem

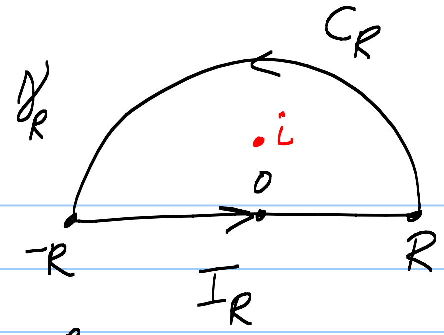
EX: $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx \quad \leftarrow \text{Fourier cosine transform}$$

Step 1: $\cos x = \operatorname{Re} e^{ix}$ $\qquad f(z) = \dfrac{e^{iz}}{z^2+1}$ $\qquad \dfrac{\cos x}{x^2+1} = \operatorname{Re} f(x)$, $x \in \mathbb{R}$

Step 2: $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx = \underbrace{\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx}_{I} + i \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{\sin x}{x^2+1}}_{\text{odd}}\, dx}_{0}$$

Step 3:

$$f(z) = \frac{e^{iz}}{z^2+1}$$



$I_R: \; z(t) = t, \; -R \le t \le R$

$C_R: \; z(t) = Re^{it}, \; 0 \le t \le \pi$

$$\int_{I_R} f(z)\,dz = \int_{-R}^{R} \frac{e^{it}}{t^2+1} \Big[\underbrace{1}_{z'(t)} \Big] \, dt = \int_{-R}^{R} \frac{\cos t}{t^2+1}\,dx \leftarrow \text{want!}$$

$$\int_{C_R}: \quad e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}\underbrace{e^{ix}}_{\text{modulus} = 1}$$

$$|e^{iz}| = e^{-y} \le 1 \text{ when } y \ge 0! \leftarrow \text{nice on upper half plane!}$$

$$\left| \int_{C_R} f\,dz \right| \le \left(\max_{C_R} \left| \frac{e^{iz}}{z^2+1} \right| \right) \underbrace{\text{Length}(C_R)}_{\pi R} \leftarrow \text{Basic estimate}$$

Denominator estimate: $\left| z^2+1 \right| = \left| z^2 - (-1) \right| \ge \Big| \underbrace{|z^2|}_{|z|=R \text{ on } C_R} - |-1| \Big|$

$$|z^2+1| \ge |R^2-1| = R^2-1 \text{ if } R > 1.$$

$$\left| \int_{C_R} f\,dz \right| \underset{\text{if } R>1}{\le} \frac{1}{R^2-1} \cdot \pi R \longrightarrow 0 \text{ as } R \to \infty$$

Step 4: Use the Residue theorem:

$$\int_{\gamma_R} f\,dz = 2\pi i \sum_{\substack{\text{poles} \\ \text{inside } \gamma_R}} \operatorname{Res} f = 2\pi i \operatorname{Res}_i \frac{e^{iz}}{z^2+1}$$

$$\int_{I_R} f\,dz + \int_{C_R} f\,dz =$$

$$\underset{\downarrow}{I} + \underset{\downarrow}{0} = 2\pi i \operatorname{Res}_i f \qquad \text{done!}$$

Step 5: Compute the residue:

$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z-i)(z+i)} = \frac{1}{z-i}\Bigg[\underbrace{\frac{e^{iz}}{z+i}}_{A_0 + A_1(z-i) + \cdots \,=\, G(z)}\Bigg] \leftarrow \text{analytic near } i$$

$$= \frac{A_0}{z-i} + A_1 + A_2(z-i) + \cdots \quad \leftarrow \text{residue!}$$

Aha! $\operatorname{Res}_i f = A_0 = \dfrac{e^{i\cdot i}}{(i)+i} \leftarrow \dfrac{G^{(0)}(i)}{0!} = G(i)$ by Taylor's,

$$= \frac{e^{-1}}{2i}$$

Answer: $I = 2\pi i \dfrac{e^{-1}}{2i} = \dfrac{\pi}{e}$

Fact: Suppose f and g are analytic near z_0 and suppose g has simple zero at z_0.

↑ zero of order 1: $g(z_0) = 0$, but $g'(z_0) \neq 0$.

Then $\operatorname{Res}_{z_0} \dfrac{f}{g} = \dfrac{f(z_0)}{g'(z_0)}$

EX: $\dfrac{e^{iz}}{z^2+1}$ $\leftarrow f(z) = e^{iz}$, $\leftarrow g(z) = z^2+1$, $g'(z) = 2z$ $\qquad g(i) = 0$, $g'(i) = 2i \neq 0$

$$\operatorname{Res}_i \frac{e^{iz}}{z^2+1} = \frac{f(i)}{g'(i)} = \frac{e^{i\cdot i}}{2i} = \frac{e^{-1}}{2i} \checkmark$$

Pf of fact: $g(z) = \underset{\overset{\shortparallel}{0}}{A_0} + \underset{\overset{\shortparallel}{g'(z_0) \neq 0}}{A_1}(z-z_0) + A_2(z-z_0)^2 + \cdots$

$$= (z-z_0)\Big[\underbrace{A_1 + A_2(z-z_0) + \cdots}_{G(z)}\Big]$$

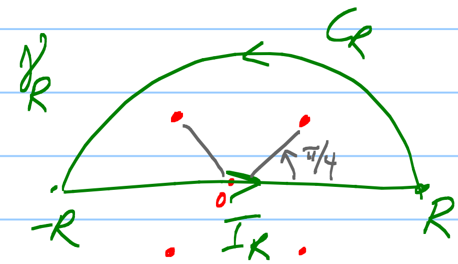
$G(z_0)$ ↑

$$\frac{f(z)}{g(z)} = \frac{1}{z-z_0}\left[\underbrace{\frac{f(z)}{G(z)}}_{B_0 + B_1(z-z_0) + \cdots}\right] \leftarrow \text{analytic near } z_0$$

$$= \frac{B_0}{z-z_0} + B_1 + B_2(z-z_0) + \cdots$$

$$\operatorname{Res}_{z_0}\frac{f}{g} = B_0 = \frac{f(z_0)}{G(z_0)} = \frac{f(z_0)}{A_1} = \frac{f(z_0)}{g'(z_0)}$$

EX: Find $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



$$f(z) = \frac{1}{z^4+1} \quad \begin{matrix} \leftarrow F(z) = 1 \\ \leftarrow G(z) = z^4+1 \\ G'(z) = 4z^3 \end{matrix}$$

Fourth roots of -1:

$$\pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i = e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)} \quad k = 0,1,2,3$$

$$\int_{I_R} f\,dz + \int_{C_R} f\,dz = 2\pi i \sum_{\substack{\text{inside}\\ \gamma_R}} \operatorname{Res} f = 2\pi i\left(\operatorname{Res}_{e^{i\pi/4}} f + \operatorname{Res}_{e^{i3\pi/4}} f\right)$$

Let $R \to \infty$:

$$I + 0 = 2\pi i\left[\frac{1}{4(e^{i\pi/4})^3} + \frac{1}{4(e^{i3\pi/4})^3}\right]$$

$$= \frac{\pi i}{2}\left[e^{-i3\pi/4} + e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{2}\left[\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right] = \frac{\pi}{\sqrt{2}}$$

ans!

Check:

$$\left|\int_{C_R} \frac{1}{z^4+1}\,dz\right| \le \underbrace{\left(\max_{C_R} \frac{1}{|z^4+1|}\right)}_{\le \frac{1}{R^4-1} \text{ if } R>1} \underbrace{\operatorname{Length}(C_R)}_{\pi R} \to 0 \text{ as } R \to \infty.$$

Book:
$$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$
$$= \frac{1}{(z-e^{i\pi/4})}\left[\frac{1}{(z-r_2)(z-r_3)(z-r_4)}\right]$$
$$\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} = \frac{1}{(e^{i\pi/4}-e^{i3\pi/4})(e^{i\pi/4}-e^{i5\pi/4})(e^{i\pi/4}-e^{i7\pi/4})} \leftarrow \text{Ugh!}$$

Fact: f analytic at z_0, then
$$\operatorname{Res}_{z_0} \frac{f(z)}{z-z_0} = f(z_0)$$