Lesson 28 $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

26, 27, 28 due Wed

Principal value integrals: $P.V. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

Principal value = Value when f is absolutely integrable.

$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

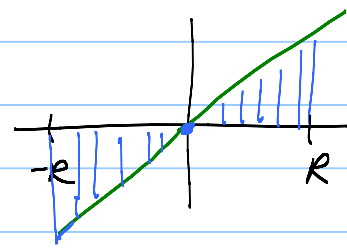
EX: $I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \leftarrow$ abs. int.

$$\left|\frac{\cos x}{x^2+1}\right| \le \frac{1}{x^2+1}$$

$$\int_{-A}^{A} \frac{1}{x^2+1}\,dx = \operatorname{ArcTan} x \Big|_{-A}^{A} \to \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$I = \lim_{R\to\infty} \int_{-R}^{R} = \lim_{\substack{A\to-\infty \\ B\to+\infty}} \int_{A}^{B}$$

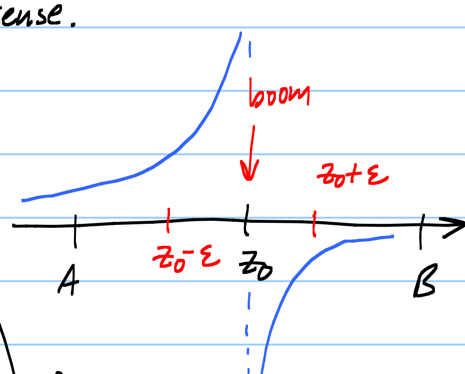
EX: $PV \int_{-\infty}^{\infty} x\,dx = 0$



Areas cancel

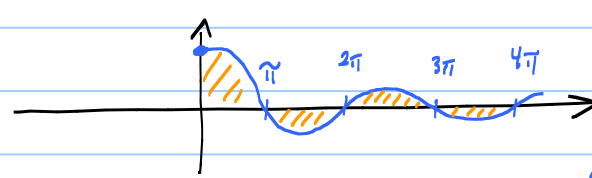
but $\int_{-\infty}^{A} x\,dx$ doesn't make sense.

Another kind of PV $\int$:



$$PV \int_{A}^{B} f\,dx = \lim_{\varepsilon\to 0} \left(\int_{A}^{z_0-\varepsilon} + \int_{z_0+\varepsilon}^{B} \right) f\,dx$$

Prob: $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$



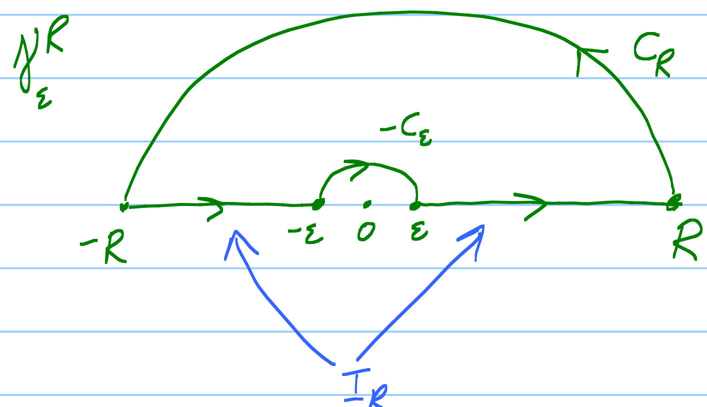
Alternating series test shows $\int_{0}^{\infty} \frac{\sin x}{x}\,dx$ exists but $\dfrac{|\sin x|}{(n+1)\pi} \le \left|\dfrac{\sin x}{x}\right| \le \dfrac{|\sin x|}{n\pi}$ when $x \in [n\pi, (n+1)\pi]$

$$\int_{\pi n}^{\pi(n+1)} : \quad \frac{2}{(n+1)\pi} \le \left| \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx \right| \le \frac{2}{n\pi}$$

Ouch! $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx \sim \frac{2}{\pi}\sum \frac{1}{n+1} \leftarrow$ divergent harmonic series

$\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally integrable. (Not absolutely int.)

Method: $\dfrac{\sin x}{x} = \operatorname{Im} \dfrac{e^{ix}}{x}$ $\quad \boxed{f(z) = \dfrac{e^{iz}}{z}}$



Cauchy Theorem says $\displaystyle\int_{\gamma_\varepsilon^R} \frac{e^{iz}}{z}\,dz = 0$

$I_R: \; z(t) = t, \quad z'(t) = 1$

$$\int_{I_R} f(z)\,dz = \left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R\right) \frac{e^{it}}{t} \cdot 1\,dt$$

$$= \left(\int_{-R}^{-\varepsilon} + \int_\varepsilon^R\right) \underset{\uparrow \text{ odd}}{\frac{\cos t}{t}} + i \underset{\uparrow \text{ even}}{\frac{\sin t}{t}}\,dt = 2i \underbrace{\int_\varepsilon^R \frac{\sin t}{t}\,dt}_{\text{want!}}$$

Next: Show $\displaystyle\int_{C_R} f(z)\,dz \to 0$

Basic estimate: $\displaystyle\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \le \underbrace{\left(\max_{C_R} \left|\frac{e^{iz}}{z}\right|\right)}_{\le \frac{1}{R}} \pi R \le \pi$

$|e^{iz}| \le 1$ on UHP

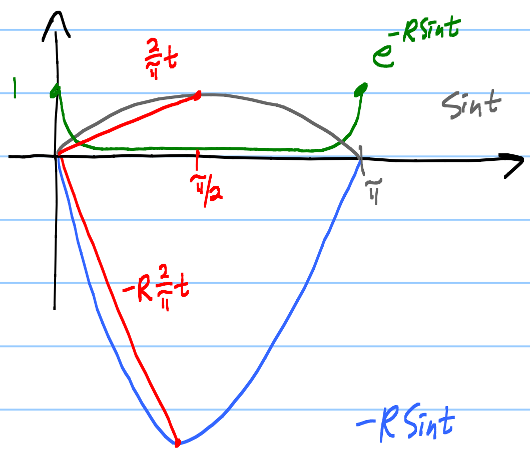
$\uparrow$ Ouch! This not $\to 0$.

Jordan's lemma: $C_R: z(t) = Re^{it} \quad 0 \le t \le \pi$

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\, iRe^{it}\,dt\right| \le \int_0^{\pi} \left|e^{iRe^{it}}\right| dt = \int_0^{\pi} e^{-R\sin t}\,dt$$

$$e^{iRe^{it}} = e^{i(R\cos t + iR\sin t)} = e^{-R\sin t}\,\underbrace{e^{iR\cos t}}_{\text{modulus } = 1}$$

$$\left|e^{iRe^{it}}\right| = e^{-R\sin t}$$



$-R\sin t \le -R\frac{2}{\pi}t$ on $[0, \frac{\pi}{2})$

$e^{-R\sin t} \le e^{-\frac{2R}{\pi}t}$ on $[0, \frac{\pi}{2}]$

So $\displaystyle\int_0^{\pi} e^{-R\sin t}\,dt \underset{\text{by symmetry}}{=} 2\int_0^{\pi/2} e^{-R\sin t}\,dt$

$$\le \underbrace{2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt}_{2\left[-\frac{\pi}{2R}e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}}$$

$-\frac{\pi}{R}\left[e^{-R} - 1\right] = \frac{\pi}{R}(1 - e^{-R})$

$\le \frac{\pi}{R} \to 0$ as $R \to \infty$

Last step: $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz \to ?$

Key: $\displaystyle\frac{e^{iz}}{z} = \frac{1 + iz + \frac{(iz)^2}{2!} + \cdots}{z}$

$$= \frac{1}{z} + \underbrace{i - \frac{z}{2!} + i\frac{z^2}{3!} - \cdots}_{g(z) \text{ entire!}}$$

So $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = \underbrace{\int_{-C_\varepsilon} \frac{1}{z}\,dz}_{-\int_{C_\varepsilon} \frac{1}{z}\,dz} + \underbrace{\int_{-C_\varepsilon} g(z)\,dz}_{}$

$$\left|\int_{-C_\varepsilon} g\,dz\right| \le \underbrace{\left(\max_{C_\varepsilon} |g|\right)}_{\le \max_{D_1(0)} |g| \text{ if } \varepsilon < 1} \pi\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$= -\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt = -i \int_0^{\pi} 1\, dt = \underline{\underline{-\pi i}}$$

$$0 = \int_{\gamma_\varepsilon^R} f(z)\, dz = \int_{I_R} \frac{e^{iz}}{z}\, dz + \int_{C_R} \frac{e^{iz}}{z}\, dz + \int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz$$

Let $R \to \infty$.
Let $\varepsilon \to 0$.

$$0 = 2i \int_0^\infty \frac{\sin t}{t}\, dt + 0 + (-i\pi)$$

So $\displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ and $\displaystyle\int_{-\infty}^\infty \frac{\sin t}{t}\, dt = \pi$

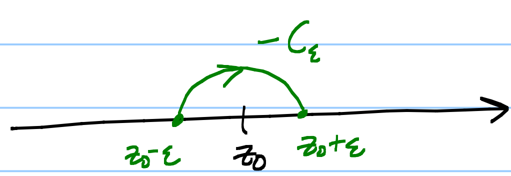
<u>Jordan's Lemma</u> used here.

Know $\displaystyle\int_{-\infty}^\infty \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \left(\text{Sum residues in UHP } \frac{P}{Q} e^{iz}\right)$

if Q has no zeroes on $\mathbb{R}$
and $\underline{\deg Q \geq \deg P + 2}$

Jordan's Lemma lets us improve this to $\deg Q \geq \deg P + \underline{\underline{1}}$

<u>General Fact</u>: Suppose $f(z)$ has a <u>simple</u> pole at $z_0 \in \mathbb{R}$.



$$\int_{-C_\varepsilon} f(z)\, dz \longrightarrow \frac{1}{2}\left(-2\pi i \operatorname{Res}_{z_0} f\right) = -\pi i \operatorname{Res}_{z_0} f$$