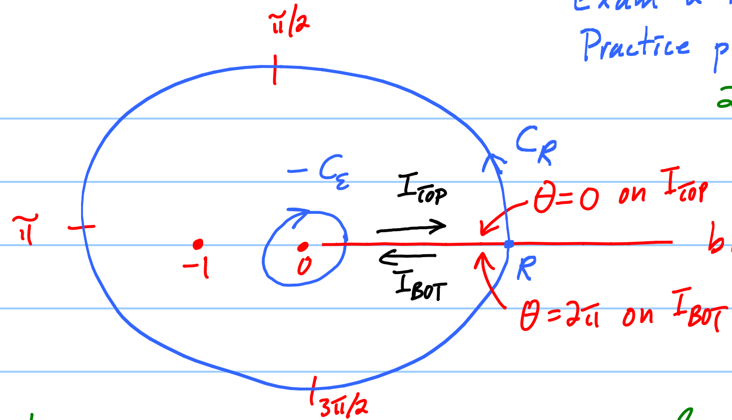
Lesson 29 Argument Principle

26, 27, 28 due Wed
Exam 2 in class, Wed Nov. 8
Practice probs & 2 old exam 2's on home pg
29, 30 due Mon Nov 6
Exam 2 : Lessons 15-30

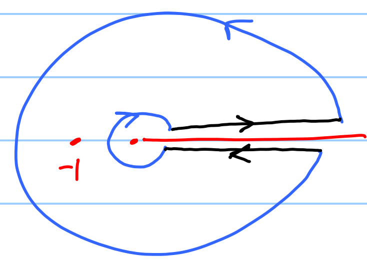


$0<\alpha<1$ $\quad \displaystyle\int_0^\infty \frac{1}{x^\alpha(x+1)}\,dx$

$$z^\alpha = e^{\alpha \log z} = e^{\alpha(\operatorname{Ln}|z| + i\theta)}$$

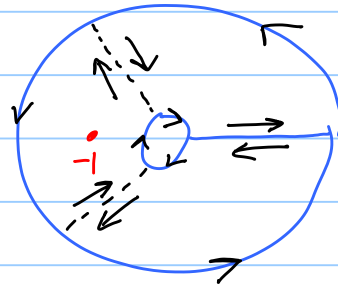
where $\theta \in \arg z$

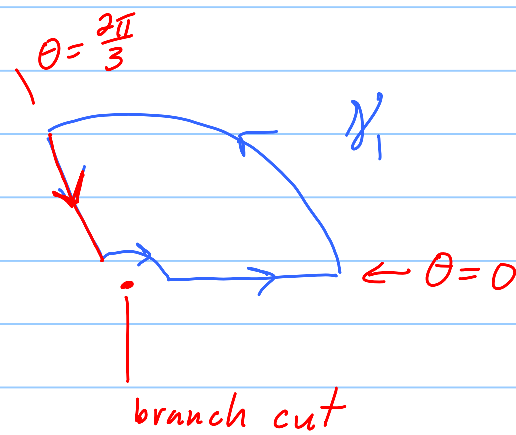
Or



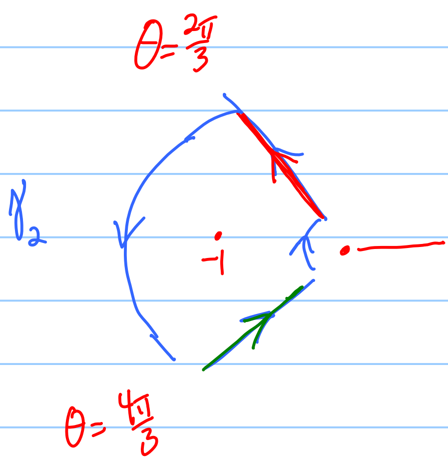
Take a limit

Or



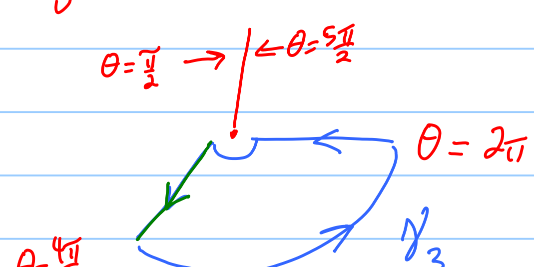


Cauchy Thm: $\displaystyle\int_{\gamma_1} f\,dz = 0$



Residue Thm: $\displaystyle\int_{\gamma_2} f\,dz = 2\pi i \operatorname{Res}_{-1} f(z)$

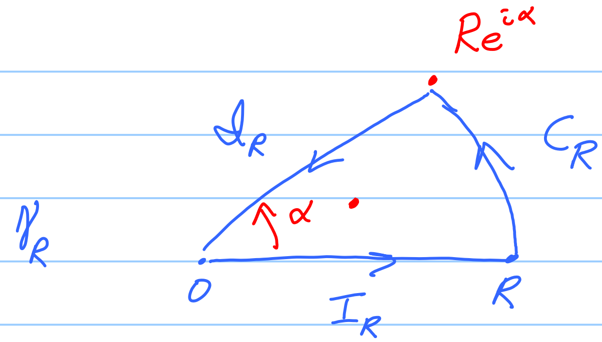
$$f(z) = \frac{1}{e^{\alpha \log z}(z+1)}$$



$\displaystyle\int_{\gamma_3} f\,dz = 0$ by Cauchy Thm.

Add them up. Get same thing.

Prob: Find $\displaystyle\int_0^\infty \frac{1}{x^5+1}\,dx$



$-\gamma_R$: $z(t) = t e^{i\alpha}$, $0 \le t \le R$

I_R : $z(t) = t$, $0 \le t \le R$

$$\int_{I_R} \underbrace{\frac{1}{z^5+1}}_{f(z)}\,dz = \int_0^R \frac{1}{t^5+1}\,(1)\,dt \quad \text{want}$$

$z'(t) = 1$

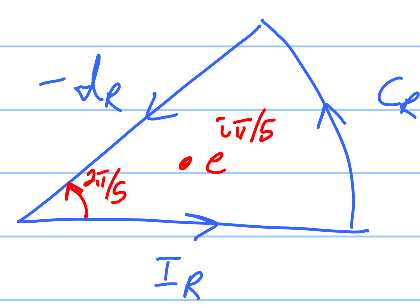
$$\int_{\gamma_R} f(z)\,dz = -\int_{-\gamma_R} f(z)\,dz = -\int_0^R \frac{1}{(\underbrace{te^{i\alpha}}_{z(t)})^5+1}\,\underbrace{(e^{i\alpha})}_{z'(t)}\,dt$$

Aha! Need $5\alpha = 2\pi$ $\boxed{\alpha = \frac{2\pi}{5}}$

$$= -\int_0^R \frac{1}{t^5\cdot 1+1}\,e^{i2\pi/5}\,dt = -e^{i2\pi/5}\int_0^R \frac{1}{t^5+1}\,dt$$

Next: $$\left|\int_{C_R} f(z)\,dz\right| \le \underbrace{\left(\max_{C_R}\left|\frac{1}{z^5+1}\right|\right)}_{\le \frac{1}{R^5-1} \text{ if } R>1}\left(\frac{2\pi}{5}R\right) \le \frac{2\pi}{5}\,\frac{R}{R^5-1} \quad \text{if } R>1$$

$\longrightarrow 0$ as $R\to\infty$.



$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{e^{i\pi/5}} f(z)$$

$$\left(\int_{I_R} + \int_{C_R} + \int_{-d_R}\right) f\, dz = 2\pi i \operatorname{Res}_{e^{i\pi/5}} \frac{1}{z^5+1} \quad \begin{matrix} \leftarrow F(z)=1 \\ \leftarrow G(z)=z^5+1 \end{matrix}$$

$$\int_0^\infty \frac{1}{t^5+1}\,dt + 0 + \left(-e^{i2\pi/5}\int_0^\infty \frac{1}{t^5+1}\,dt\right) = 2\pi i \frac{F(e^{i\pi/5})}{G'(e^{i\pi/5})}$$

$G(e^{i\pi/5}) = 0$

$G'(z) = 5z^4$

$G'(e^{i\pi/5}) = 5e^{i4\pi/5}$

$\uparrow$ not zero!

$$(1-e^{i2\pi/5})\, I = 2\pi i \frac{1}{5e^{i4\pi/5}}$$

$$I = 2\pi i \frac{1}{5e^{i4\pi/5}(1-e^{i2\pi/5})} \cdot \frac{e^{-i\pi/5}}{e^{-i\pi/5}}$$

$$= \frac{\pi}{5} \frac{2i}{(e^{-i\pi/5} - e^{i\pi/5})} \cdot \underbrace{e^{-i4\pi/5} \cdot e^{-i\pi/5}}_{e^{-i\pi} = -1}$$

$$= \frac{\pi}{5} \frac{2i}{e^{i\pi/5} - e^{-i\pi/5}} = \frac{\pi/5}{\sin \pi/5} \quad \text{ans}$$

Argument Principle: $f(z)$ has a zero of order N at z_0:

$$f(z) = (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots\right]}_{F(z)} = (z-z_0)^N F(z)$$

where $F(z_0) = a_N \neq 0$

Fact: $\operatorname{Res}_{z_0} \dfrac{f'(z)}{f(z)} = N \leftarrow$ order of zero of f at z_0 when f has a zero at z_0

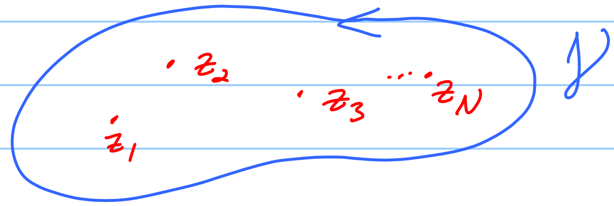
Why:
$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}F(z) + (z-z_0)^N F'(z)}{(z-z_0)^N F(z)}$$

$$= \frac{\overset{\operatorname{Res}_{z_0}}{N}}{z-z_0} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } z_0}$$

Fact 2: Repeat using $-N$ in place of N.

$$\operatorname{Res}_{z_0} \frac{f'(z)}{f(z)} = -N \longleftarrow = \text{minus the order of the pole of } f \text{ at } z_0 \text{ when } f \text{ has a pole at } z_0$$

Argument Principle I:



Suppose f is analytic inside and on a simple closed curve γ and suppose f has no zeroes on γ.

Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\text{Total number of zeroes of } f \\ \text{inside } \gamma \\ \text{counted with multiplicity}\end{array}\right)$$

Why: $\operatorname{Res}_{z_k} = \left(\text{Order of zero of } f \text{ at } z_k\right)$.

Arg. Princ. II:

Same hyp, but allow f to have zeroes and poles inside γ.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\text{Total \# zeroes } f \\ \text{inside } \gamma \\ \text{counted with mult.}\end{array}\right) - \left(\begin{array}{c}\text{Total \# poles } f \\ \text{inside } \gamma \\ \text{counted with mult.}\end{array}\right)$$