Lesson 30 Rouché's Theorem

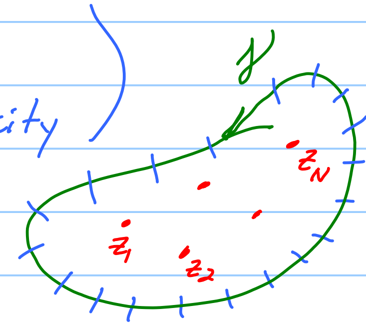
29, 30 due Monday
Exam 2 on Wed in class

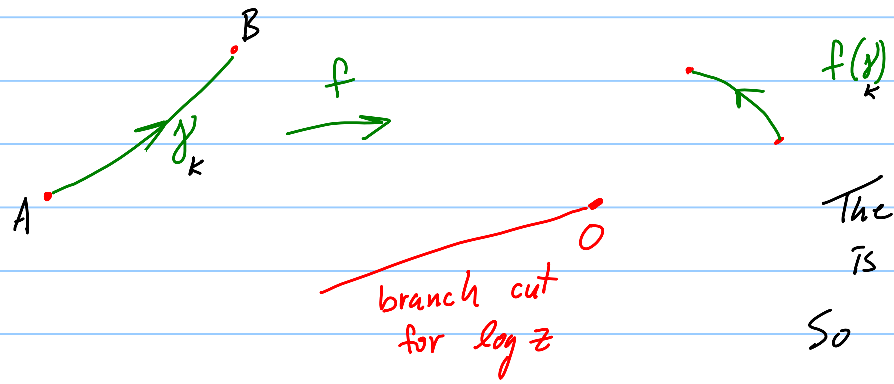
Argument Principle I: f analytic inside and on simple closed γ in counterclockwise direction, f non-vanishing on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f \text{ inside } \gamma \\ \text{counted with multiplicity}\end{array}\right)$$

$$= \sum_{k=1}^{N} m_k$$

where m_k = mult zero at z_k





Then this branch of log is analytic on $f(\gamma)$.

So $\frac{d}{dz}\left(\log f(z)\right) = \frac{f'(z)}{f(z)}$ on segment of γ.

Now $$\int_{\gamma_k} \frac{f'}{f}\,dz = \int_{\gamma_k} \frac{d}{dz}\left(\log f\right) dz = \log f(B) - \log f(A)$$

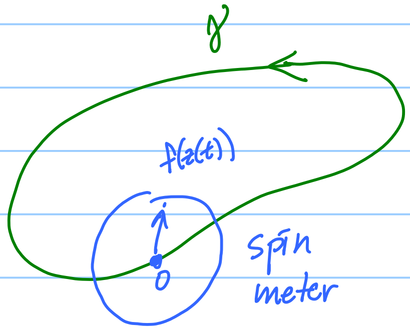
$$= \left(\operatorname{Ln}|f(B)| + i \arg f(B)\right) - \left(\operatorname{Ln}|f(A)| + i \arg f(A)\right)$$

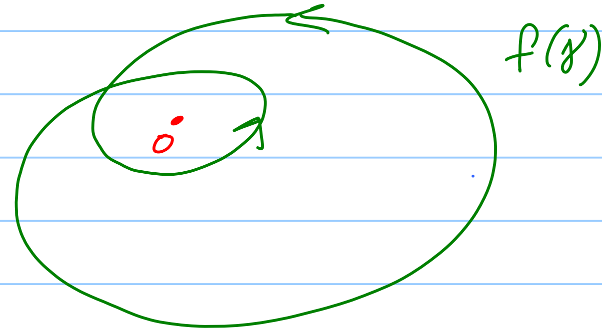
$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i \underset{\gamma_A^B}{\Delta} \arg f(z)$$

Aha! Add these up going around γ. The real parts pairwise cancel and the starting point cancels the ending point.

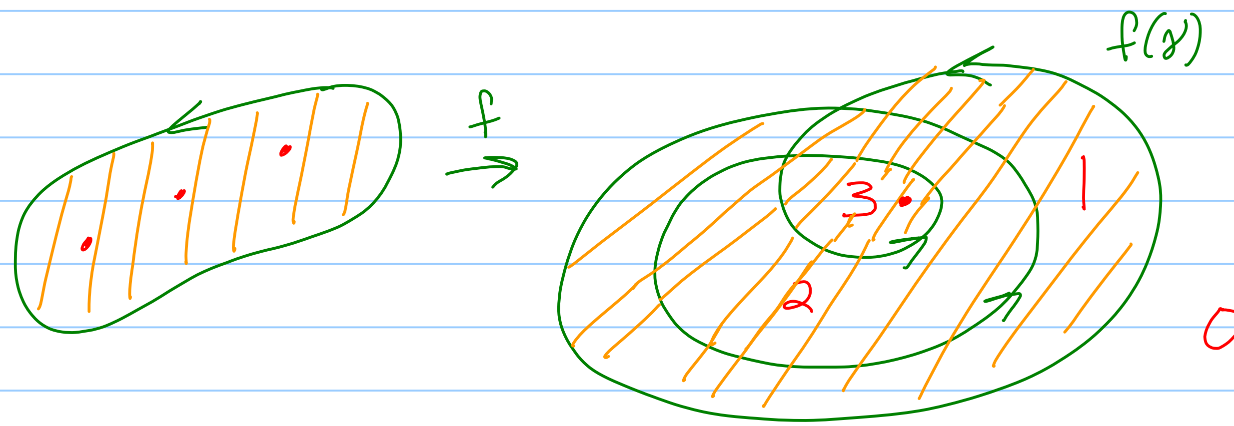
Get $$\int_\gamma \frac{f'}{f}\,dz = i \sum \underset{\gamma_k}{\Delta} \arg f(z) = i \underset{\gamma}{\Delta} \arg f$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{1}{2\pi}\underset{\gamma}{\Delta}\arg f = \text{\# times } f(z) \text{ spins around origin as } \gamma \text{ is traced out!}$$





EX:



Consequence: (The Open Mapping Theorem) A non-constant analytic function on a domain maps open sets to open sets.

Way false $\mathbb{R}\to\mathbb{R}$: $h(t) = 1-t^2$ maps $(-1,1)$ to $(0,1]$ ← not open

max!

Rouché's Theorem: Suppose f and h are analytic inside and on a counterclockwise simple closed curve γ and suppose

$$|h(z)| < |f(z)| \text{ on } \gamma.$$

Then $f(z)$ and $f(z)+h(z)$ have the same number of zeroes inside γ (counted with mult).

EX: $\underbrace{z^3+9z}_{h(z)} + \underbrace{27}_{f(z)}$ on $D_2(0)$

$$|z^3+9z| \le |z|^3 + 9|z| \le \underbrace{2^3+9\cdot 2}_{26} \text{ on } C_2(0)$$

$$26 < 27$$

So $f(z)$ and $f+h$ have same # zeroes inside $C_2(0)$.

no zeroes   no zeroes!

EX: Show that all the roots of $z^6 - 5z^2 + 10$ fall in $\{z : 1 < |z| < 2\}$.

On outer curve $C_2(0)$: $\quad z^6 - 5z^2 + 10$

big one
$|z^6| = 64$ on C_2

$f(z) = z^6$
$h(z) = -5z^2 + 10$
$|-5z^2 + 10| \le 30 < 64$ on C_2

So $f(z) = z^6$ and $f+h = z^6 - 5z^2 + 10$ have same # zeroes in C_2, namely $\underline{\underline{6}}$.

Know: All 6 zeroes of six deg poly fall in $C_2(0)$.

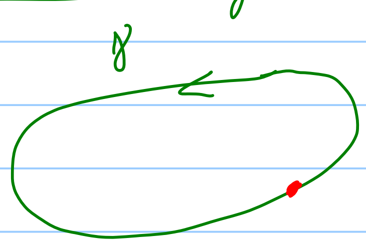
On inner circle $C_1(0)$: $\quad \underbrace{z^6 - 5z^2}_{h} + 10$

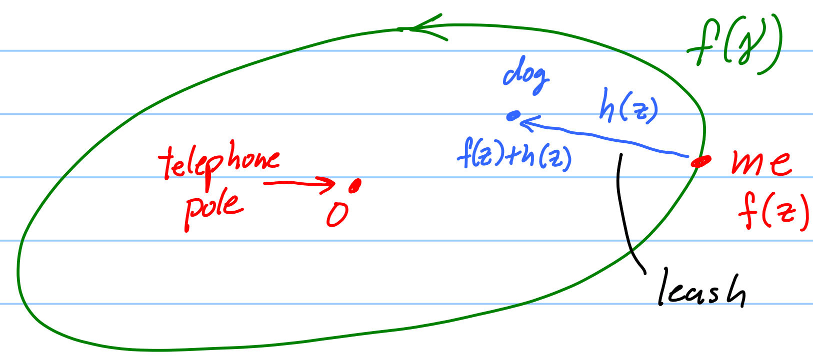
big one
$f(z) = 10$

$|h(z)| \le 1^6 + 5 \cdot 1^2 = 6 < 10$

f has no zeroes in $C_1(0)$. Hence, our poly $f+h$ has no zeroes there.

Geometric meaning of Rouché:





$|h(z)| < |f(z)|$

(leash length) < (distance from me to pole)

So dog goes around pole same number of times I do!

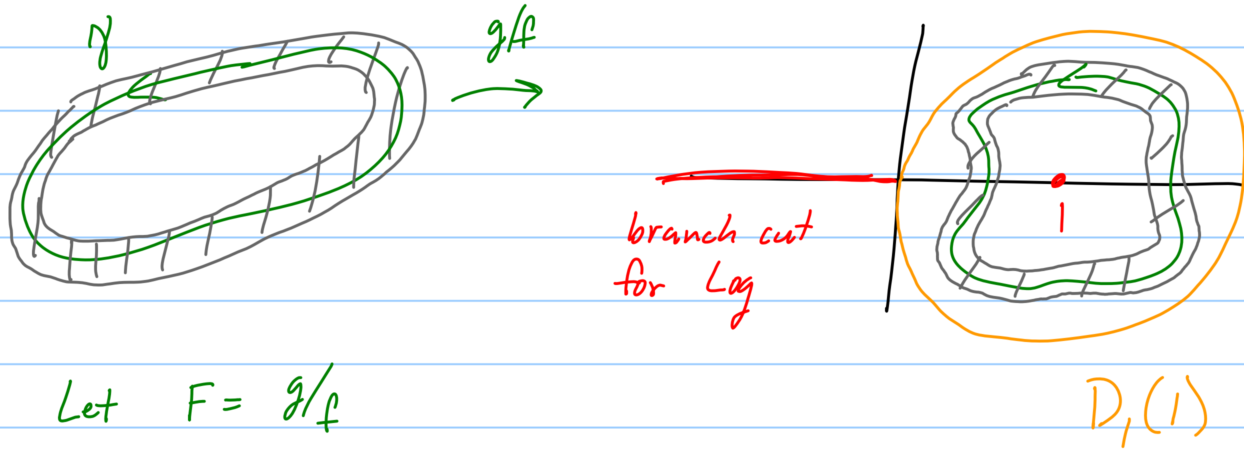
Why Rouché's works:

$|h(z)| < |f(z)|$ on γ. ← see that f has no zeroes on γ.

Let $g = f + h$. Then $h = g - f$.

$|g(z) - f(z)| < |f(z)|$ on γ. ← Divide by f.

$$\left| \frac{g(z)}{f(z)} - 1 \right| < 1 \quad \text{on } \gamma.$$



Log analytic there.

Let $F = g/f$

$$\underbrace{\frac{F'}{F}}_{\frac{d}{dz}(\operatorname{Log} F)} = \frac{g'}{g} - \frac{f'}{f}$$

Integrate around γ and use arg princ.

$$\underbrace{\frac{1}{2\pi i} \int_\gamma \frac{d}{dz} \operatorname{Log} F \, dz}_{0} = \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{g'}{g} \, dz}_{\#\text{ zeroes } g} - \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} \, dz}_{\#\text{ zeroes } f}$$