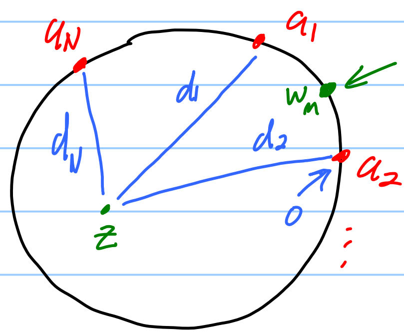
Review for Exam 2

Exam 2 on Wed in class

Office Hour Tues 1:00 – 2:30

Prac. Prob. 6:



$|f(w_m)| = M > 1$. $M = \underset{D_1(0)}{\mathrm{Max}} |f|$

$D(z) = \prod_{k=1}^{N} d_k$

Show $D = 1$ at some point on $C_1(0)$.

Hint: $f(z) = \prod_{k=1}^{N} (z - a_k)$ ⟵ analytic and $|f(z)| = \prod_1^N \underbrace{|z-a_k|}_{d_k} = D(z)$

$|f(0)| = \prod_1^N |a_k| = 1.$

Note: f is not constant. Max Modulus Princ ⟹ 1 is not the max of $|f|$ on $D_1(0)$. MMP 2 says $|f|$ attains its max on $C_1(0)$. There is a point w_m on $C_1(0)$ where $|f(w_m)| = M > 1$,

Aha! $f(a_k) = 0$ for each a_k, $f(e^{i\theta})$ is continuous. Intermediate Value Theorem shows there is a point on $C_1(0)$ between any a_k and w_m where $|f| = 1$.

Prob: $\displaystyle\sum_{n=0}^{\infty} \overbrace{\frac{n^n}{n!\,2^n} \underbrace{z^{2n}}_{\text{harder} = z^{n^2}}}^{u_n}$ Ratio Test

$$\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{\dfrac{(n+1)^{(n+1)}}{(n+1)!\,2^{n+1}} z^{2(n+1)}}{\dfrac{n^n}{n!\,2^n} z^{2n}} \right| = \frac{1}{n+1} \cdot \frac{(n+1)^n \cdot (n+1)}{n^n} \cdot \frac{1}{2} \cdot |z|^2$$

$$= \frac{1}{2}\left(1 + \frac{1}{n}\right)^n |z|^2 \longrightarrow \frac{e}{2}|z|^2 \quad \begin{matrix} < 1 & \text{conv} \\ > 1 & \text{div} \end{matrix}$$

Conv if $|z|^2 < \frac{2}{e}$ $\quad |z| < \sqrt{\frac{2}{e}}$ ⟵ R

Harder Prob: $\left| \dfrac{\frac{(n+1)^{n+1}}{(n+1)!} z^{(n+1)^2}}{\frac{n^n}{n!} z^{n^2}} \right| = \underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow \; e} |z|^{2n+1}$

Hmmm, If $|z|<1$, then $\left|\frac{u_{n+1}}{u_n}\right| \to 0$, and so series conv,

Conclude $R \geq 1$.

If $|z|>1$, $\left|\frac{u_{n+1}}{u_n}\right| \to \infty$, and series diverges.

So $R = 1$.

Prob. 10: $a_n \to 0$ as $n \to \infty$. Show R. of. C for

$\sum_0^\infty a_n z^n$ is $R \geq 1$.

Hadamard's: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \sup_{n\geq N} \sqrt[n]{|a_n|}$

$0 \leq |a_n| < 1$ if $n > N_0$ since $a_n \to 0$.

Now take n-th root: $0 \leq \sqrt[n]{|a_n|} < 1$. Aha! Lim of Sups has to be ≤ 1.

So $\frac{1}{R} \leq 1$.

And $R \geq 1$. ✓

Or better yet, since $a_n \to 0$, it is a bounded seq.

So $\exists$ M such that $|a_n| \leq M$ for all n.

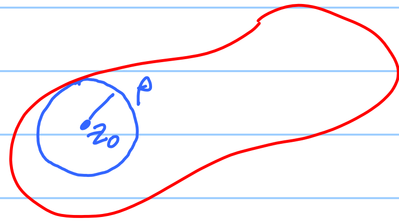
Now, if $|z|<1$, then $|a_n z^n| = |a_n||z|^n \leq \underbrace{M|z|^n}_{\text{Conv. Geom Series}}$

Comparison test shows $\sum a_n z^n$ conv if $|z|<1$. So $R \geq 1$.

Prob 4: $f(z) = \frac{\sin z}{z}$ $\quad z = \pi$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\pi)^n \qquad a_n = \frac{f^{(n)}(\pi)}{n!}$$

Big Fact:



f analytic on Ω

R of C about $z_0 \geq \rho$

Know: $z=0$ is a removable sing for $\frac{\sin z}{z}$. Set

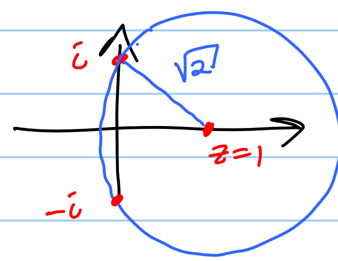
$$F(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is entire! R. of C. for F about $z=\pi$ is ∞.

Aha! F and f have the same power series about $z=\pi$.

so $R=\infty$.

EX: $f(z) = \frac{1}{z^2+1}$ $\quad z = 1$.



$R \geq \sqrt{2}$

Poles at $\pm i$ show $R \leq \sqrt{2}$.

so $R = \sqrt{2}$

Prob: $I = \displaystyle\int_0^{2\pi} \frac{1}{(2+\sin\theta)^2}\, d\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$$

$$= \frac{z(\theta) - \frac{1}{z(\theta)}}{2i}$$

$C_1(0)$: $z(\theta) = e^{i\theta}$

$z'(\theta) = ie^{i\theta}$

$dz = z'(\theta)\, d\theta = ie^{i\theta}\, d\theta$

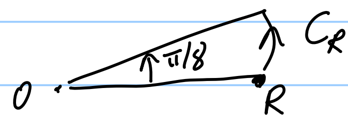
$$I = \int_0^{2\pi} \frac{1}{\left(2 + \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}\right)^2} \frac{ie^{i\theta}\, d\theta}{ie^{i\theta}}$$

dz

$ie^{i\theta} \leftarrow iz(\theta)$

$$= \int_{C_1(0)} \frac{1}{\left(2+\frac{z-\frac{1}{z}}{2i}\right)^2}\cdot\frac{1}{iz}\,dz = 2\pi i \sum_{\text{poles } \underline{\text{inside}}\ C_1(0)} \text{Res}$$

<u>Prob</u>:



$$\left|\int_{C_R} e^{-z^2}\,dz\right| \le \left(\max_{C_R}\left|e^{-z^2}\right|\right)\cdot\left(\frac{\pi}{8}R\right)$$

Ouch! Basic Estimate does <u>not</u> show $\int_{C_R} \to 0$.

Parametrize C_R: $z(t)=Re^{it}$, $0\le t\le \frac{\pi}{8}$.

Write out $\left|\int_{C_R}\right| = \left|\int_0^{\pi/8}\right| \le \int_0^{\pi/8} \underbrace{\left|(\qquad)\right|}_{\text{Jordan estimate idea.}}\,dt$