Lesson 35 7.3 Linear Fractional Transformations (LFTs)

Lessons 32, 35, 36 due Wed, Nov 29

Def^n: $L(z) = \dfrac{az+b}{cz+d}$ where $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc \neq 0$

is an LFT (book "Möbius transf.")

Note: If det = 0, then one row of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = const. the other row. Then $L(z) = \text{const.}$ Or worse, $c=d=0$ and L not defined.

Step 1: Do long division:

$$\begin{array}{r} \frac{a}{c} \\ cz+d \overline{)\, az+b} \\ az+\frac{da}{c} \\ \hline b-\frac{da}{c} \end{array}$$

$$L(z) = a/c + \frac{\left(b-\frac{da}{c}\right)}{cz+d}$$

Step 2: Basic maps:

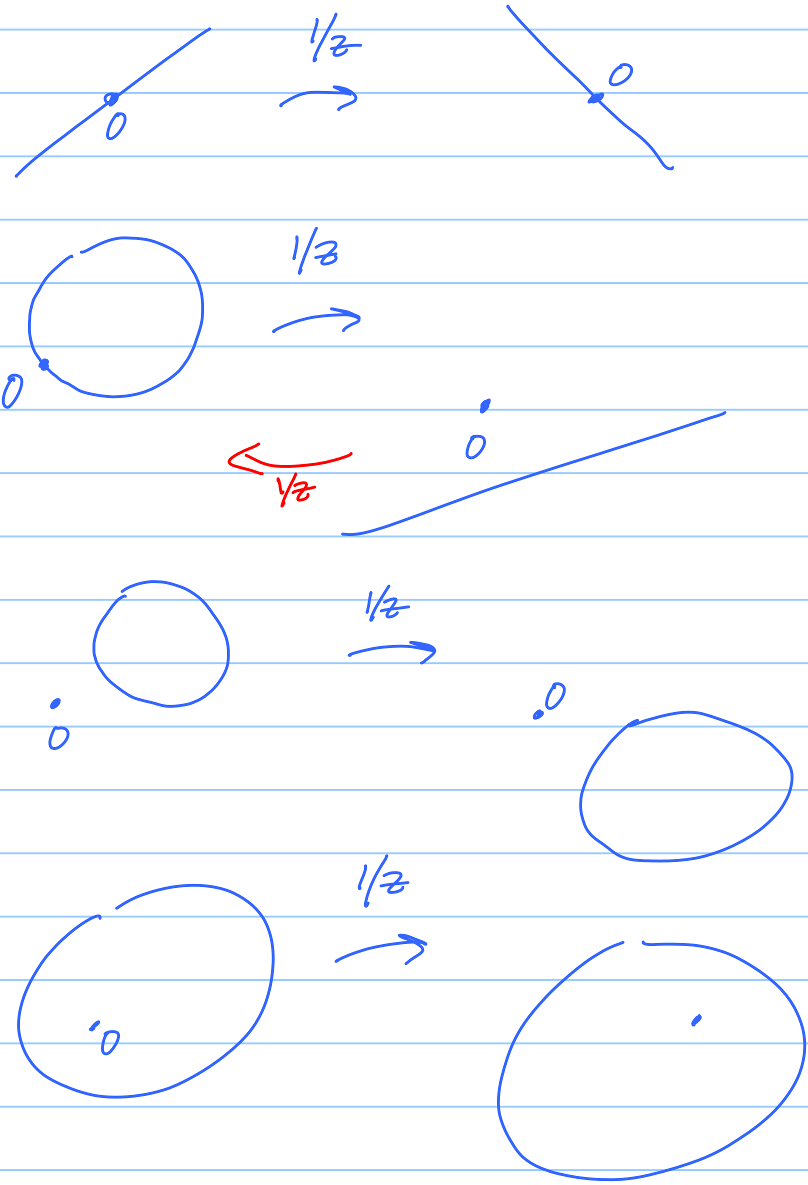
1) $z \mapsto rz$ $\quad r>0$ is real. Stretch
2) $z \mapsto e^{i\theta}z$ $\quad \theta = \text{angle}$ Rotation
3) $z \mapsto z+B$ $\quad B \in \mathbb{C}$ Translation
4) $z \mapsto 1/z$ Inversion

Fact: Basic maps generate the LFTs via composition.

Why: $L(z) = \underbrace{a/c}_{B} + \dfrac{\overbrace{\left(b-\frac{da}{c}\right)}^{Re^{i\alpha}}}{\underbrace{c}_{c=re^{i\theta}}z+d}$ (if $c \neq 0$)

7 steps: Stretch by r, rotate by θ, translate by d, invert, Stretch by R, rotate by α, translate by B.

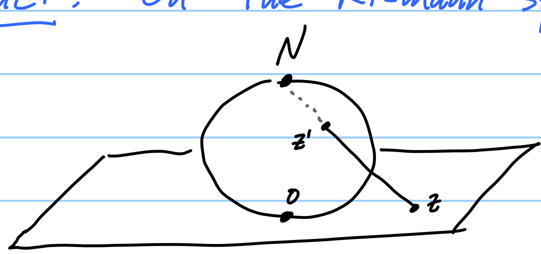
Big Fact: Maps of type 1, 2, 3 take lines to lines and circles to circles. Maps of type 4:



Fact about inversions: They take {lines, circles} to {lines, circles}

Thm: LFTs : {lines, circles} $\to$ {lines, circles}.

Nifty fact: On the Riemann sphere



{lines, circles} in $\mathbb{C}$ correspond to slices of $\hat{\mathbb{C}}$ by planes, which are circles.

Maps 1, 2, 3: $\mathbb{C} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C}$, $\infty \mapsto \infty$

Map 4: $\mathbb{C}-\{0\} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C}-\{0\}$ $\quad 0 \mapsto \infty$, $\infty \mapsto 0$

<u>Fact:</u> LFTs are 1-1 onto self maps of $\hat{\mathbb{C}}$.

<u>Case:</u> $L(z) = Az + B$ $\quad \infty \mapsto \infty$

<u>Case $c \neq 0$:</u> $\quad L(z) = \dfrac{az+b}{cz+d} = \dfrac{a + \frac{b}{z}}{c + \frac{d}{z}}$

$\lim\limits_{z \to \infty} L(z) = \dfrac{a}{c}$ $\quad$ Write $\underline{L(\infty) = \frac{a}{c}}$.

<u>And:</u> $cz + d = 0$ $\quad$ Write $\underline{L(-\frac{d}{c}) = \infty}$.

$z = -\dfrac{d}{c}$

<u>Nifty fact:</u> $\quad L(z) = \dfrac{az+b}{cz+d} \sim M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det \neq 0$

$$L_1 \circ L_2 \sim M_1 M_2$$

$$L^{-1} \sim M^{-1}$$

Finding L^{-1}: $\quad W = \dfrac{az+b}{cz+d} \leftarrow$ solve for $z =$ fcn of w.