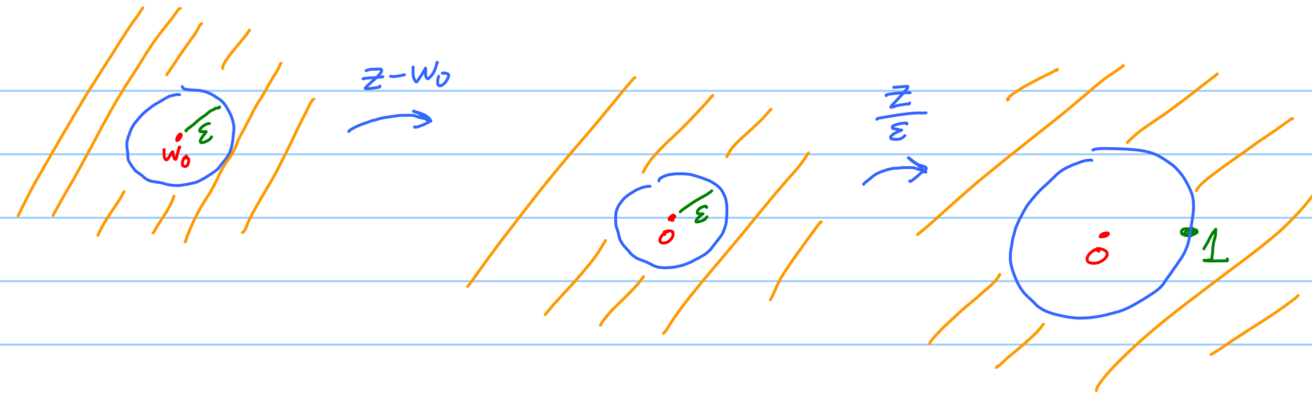
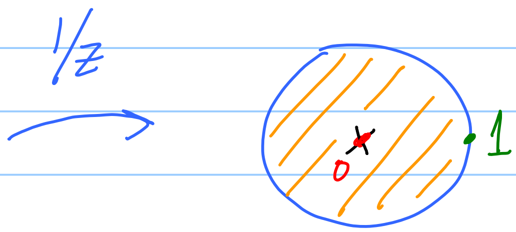
Lesson 36 cont'd LFTs   32, 35, 36 due Wed after break





$$\text{Composition} = \frac{\varepsilon}{z - w_0} = L_\varepsilon(z)$$

Fact: Suppose f has an isolated singularity at z_0. If there is an $\varepsilon > 0$ such that f is analytic on $D_\varepsilon(z_0) - \{z_0\}$ and $f(D_\varepsilon(z_0) - \{z_0\})$ misses an open set, i.e., it misses a disc $D_\rho(w_0)$. Then z_0 must be a removable sing for f or a pole.

Why: $L_\rho \circ f$ is bounded on $D_\varepsilon(z_0) - \{z_0\}$. So z_0 is removable for $\dfrac{\varepsilon}{f(z) - w_0} \equiv H(z) \leftarrow$ analytic near z_0

Now $f(z) = w_0 + \dfrac{\varepsilon}{H(z)} \leftarrow$ $H(z_0) \neq 0$, z_0 removable for f; $H(z_0) = 0$, z_0 is a pole of f.

Def^n in some books: z_0 is an essential sing for f if $f(D_\varepsilon(z_0) - \{z_0\})$ is dense in $\mathbb{C}$ no matter how small $\varepsilon > 0$ is.

Key facts about LFT's to determine mappings

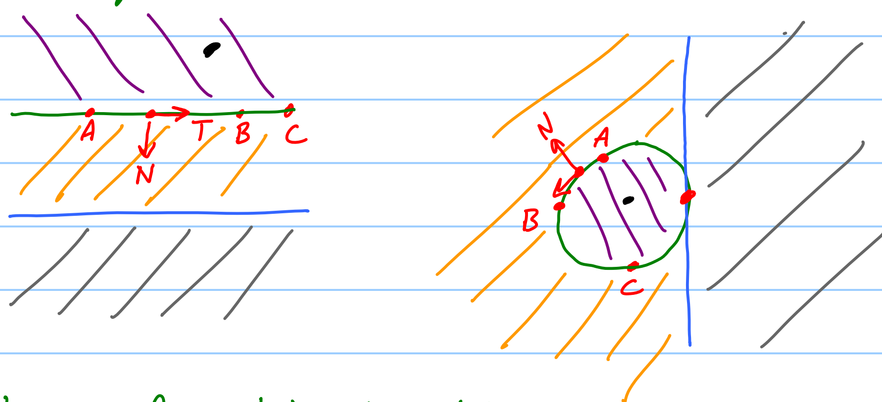
1) LFTs map $\{\text{lines, circles}\}$ to $\{\text{lines, circles}\}$.

a) 3 pts in $\mathbb{C}$ determine a circle ... or a <u>line</u> if they are colinear.

b) 2 pts plus ∞ determine a line

2) LFTs : $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ 1-1, onto.

Consequenty, if lines and circles cut $\mathbb{C}$ into regions, those regions each get mapped 1-1, onto regions of $\mathbb{C}$ – (images of those lines, circles).



3) Use conformality to determine where regions go.

<u>or</u> just test <u>one point</u> in each region to determine.

<u>Recall</u>: a, b, c distinct pts in $\mathbb{C}$

$$L(z) = \frac{c-b}{c-a}\,\frac{z-a}{z-b} \qquad \left.\begin{array}{l} L(a) = 0 \\ L(b) = \infty \\ L(c) = 1 \end{array}\right\}$$

3 pts determine L. so L is the uniq LFT doing this

<u>Problem</u>: Find an LFT that maps

$z_1 \longmapsto w_1$
$z_2 \longmapsto w_2$
$z_3 \longmapsto w_3$

z_j's distinct in $\mathbb{C}$
w_j's distinct in $\mathbb{C}$

circle (or line) circle (or line)

Sol^n: $L_1(z) = \dfrac{z_3-z_2}{z_3-z_1}\cdot\dfrac{z-z_1}{z-z_2}$ $\qquad L_1(z_j) = \begin{cases} 0 & \leftarrow j=1 \\ \infty & 2 \\ 1 & 3 \end{cases}$

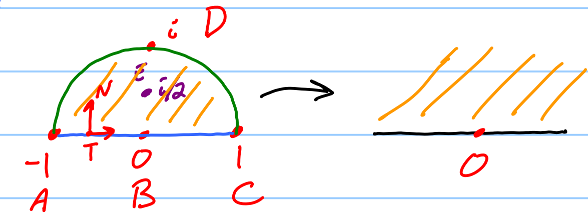
$L_2(w) = \dfrac{w_3-w_2}{w_3-w_1}\cdot\dfrac{w-w_1}{w-w_2}$ $\qquad L_2(w_j) = \begin{cases} 0 \\ \infty \\ 1 \end{cases}$

$L_2^{-1}\circ L_1$ does it!

Finding L_2^{-1} is easy: $\quad L(z) = \dfrac{Az+B}{Cz+D}$: $\quad L^{-1}(w) = \dfrac{Dw-B}{-Cw+A}$

Composition corresponds to matrix multiplication.

Prob: Find a conformal map of



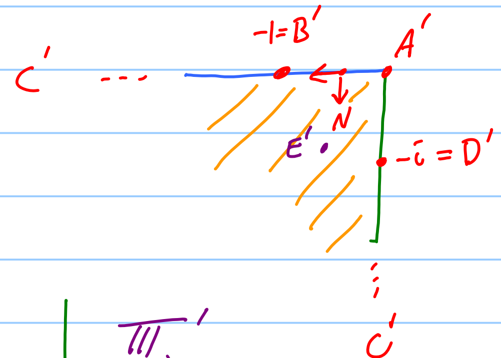
Sol^n: Hmmm. If use a LFT to blow $z=1$ out to ∞, the line and circle both go to lines. $\quad L(z) = \dfrac{z+1}{z-1}$

$L(1) = \infty$

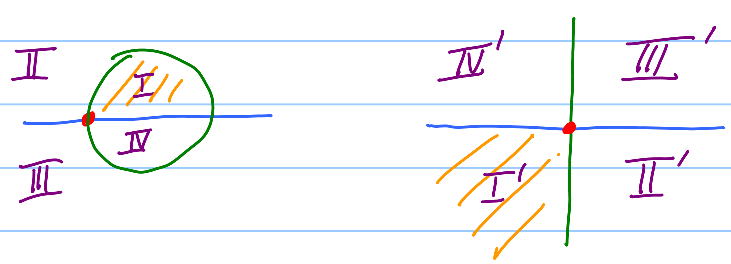
$L(-1) = 0$

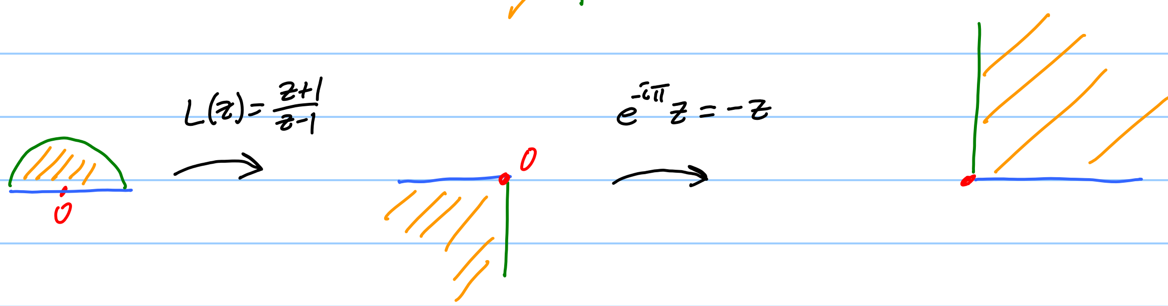
$L(0) = -1$

$L(i) = \dfrac{i+1}{i-1}\cdot\dfrac{-i-1}{-i-1} = \dfrac{-2i}{2} = -i$

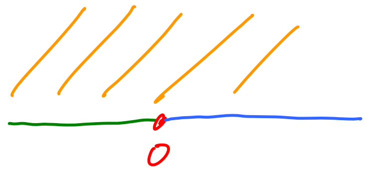


$L(\frac{i}{2}) = -\frac{3}{5} - \frac{4}{5}i$



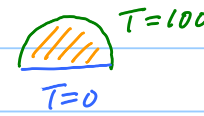


$z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$ $\longrightarrow$



Ans: $\left[-\left(\frac{z+1}{z-1}\right)\right]^2 = \left(\frac{z+1}{z-1}\right)^2$

Prob: Heat prob: Find a harmonic fcn on



Aha! $u(z) = \frac{100}{\pi} \operatorname{Arg} z$ solves prob on image.

So Sol^n is $\frac{100}{\pi} \operatorname{Arg}\left(\frac{z+1}{z-1}\right)^2$

Great combo: Riemann Mapping Theorem plus the Poisson Formula for solving Dirichlet Problem on $D_1(0)$.