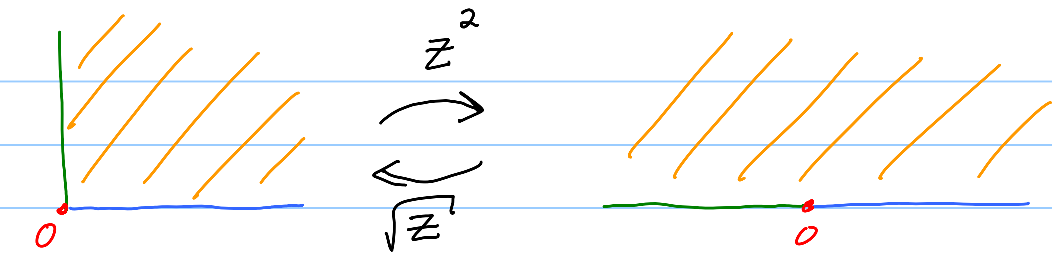
# Lesson 37 Schwarz-Christoffel maps

32, 35, 36 due next Wed.
Read 7.5




$$z^2 = \left(re^{i\theta}\right) = r^2 e^{i2\theta}$$

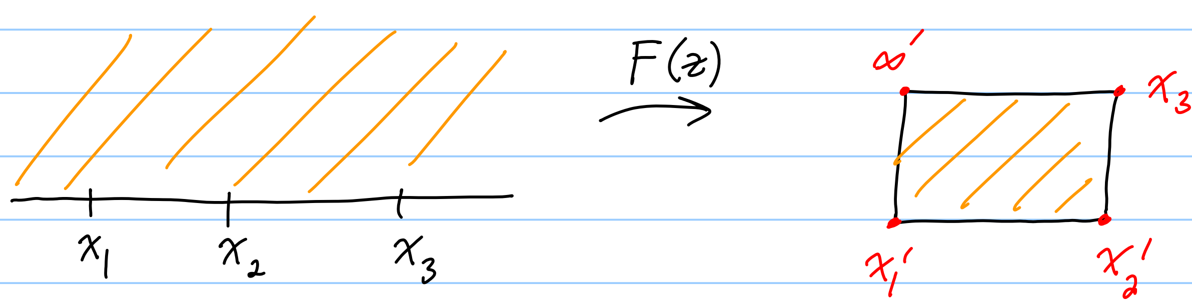
Fact: $z^2$ preserves angles where $z \neq 0$ because $\frac{d}{dz}(z^2) = 2z \neq 0$ if $z \neq 0$. But angles doubled at $z=0$,

Inverse: $\sqrt{z} = e^{\frac{1}{2}\operatorname{Log} z}$

$$\frac{d}{dz}\left(\sqrt{z}\right) = \underbrace{\left(e^{\frac{1}{2}\operatorname{Log} z}\right)}_{\sqrt{z}} \cdot \underbrace{\frac{d}{dz}\left(\frac{1}{2}\operatorname{Log} z\right)}_{\frac{1}{2}\frac{1}{z}}$$

$$= \frac{1}{2}\frac{\sqrt{z}}{z} = \frac{1}{2}\frac{e^{\frac{1}{2}\operatorname{Log} z}}{e^{\operatorname{Log} z}} = \frac{1}{2}\frac{1}{e^{\frac{1}{2}\operatorname{Log} z}} = \frac{1}{2}\frac{1}{\sqrt{z}} \checkmark$$

Prob: Find a conformal map of the UHP onto a rectangle.



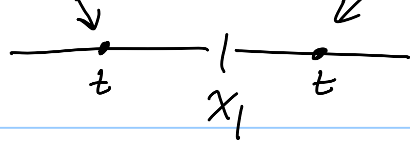

Hmmm: $F'(z) = (z - x_1)^{-1/2}$ does the right thing at $x_1$.

Think about an $F(z)$ such that

$$F'(z) = A \cdot (z - x_1)^{-1/2}(z - x_2)^{-1/2}(z - x_3)^{-1/2} + B$$

It works! Save $A, B$ for last. Use principal branches of $z^{1/2}$ in product.

$(t-x_1)^{-1/2} = \frac{1}{i}|t-x_1|^{-1/2}$ $\qquad$ $(t-x_1)^{-1/2} = |t-x_1|^{-1/2}$



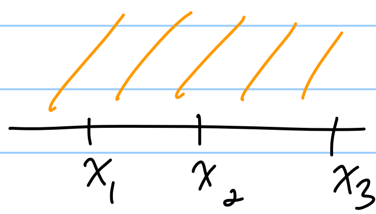

Aha! Let $t$ slide along $\mathbb{R}$. $F'(t) = \cdots$ is an ODE problem!

Aha! Direction jumps at each $x_j$ by multiplication by $i$ as I pass $x_j$. $i = e^{i\pi/2}$ ← turn left by 90°!

$$\frac{d}{dt}F(t) = u' + iv' = (\text{Complex const})\cdot |t-x_1|^{-1/2}|t-x_2|^{-1/2}|t-x_3|^{-1/2}$$

Easy, straight forward anti-differentiantion!

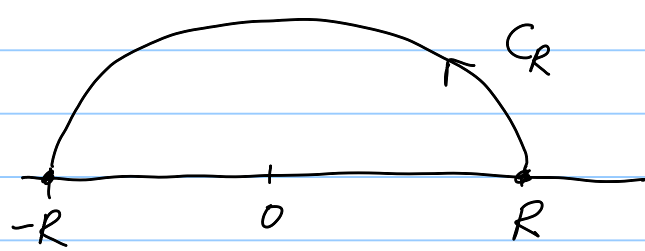
Hmmm.





Miracle! $a$ and $b$ hook up!

$$F(z) = \int_{L_i}^{z} (z-x_1)^{-1/2}(z-x_2)^{-1/2}(z-x_3)^{-1/2}\,dz$$




$$|F(-R) - F(R)| = \left|\int_{C_R} F'\,dz\right| \le \underbrace{\operatorname{Max}_{C_R} |F'|}_{\le \frac{1}{\alpha (R^3)^{1/2}}} \cdot \pi R \to 0 \text{ as } R \to \infty$$

Check that $|\sqrt{z}| = \sqrt{|z|}$.

So $$|F'(z)| = \frac{1}{\sqrt{|z-x_1||z-x_2||z-x_3|}} = \frac{1}{|\text{poly of deg } 3|^{1/2}}$$

Basic poly estimate $\alpha |z|^3 \leq |\text{deg } 3 \text{ poly}|$ for big $z$.

Estimate shows that corners hook up.

Easy to check only go a finite distance as $t \to \pm\infty$.

$t \to +\infty$ : $$|F'(t)| \leq \frac{1}{(\alpha t^3)^{1/2}}$$ ← convergent integral as $t \to \infty$.

Why does inside map one-to-one onto the inside?

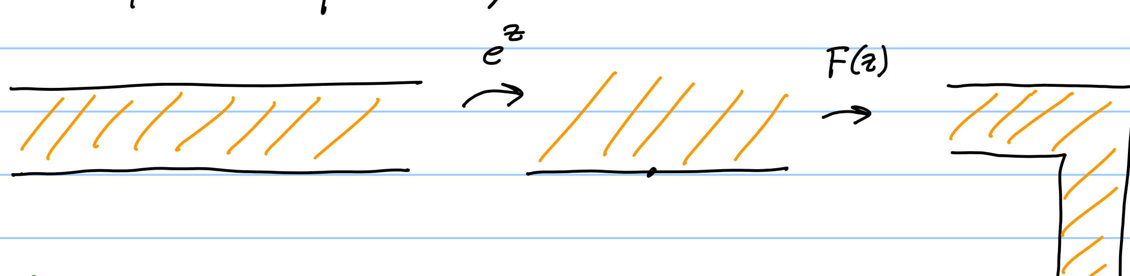
Argument principal.

spin around exactly once as image of dot crawls along $\mathbb{R}$

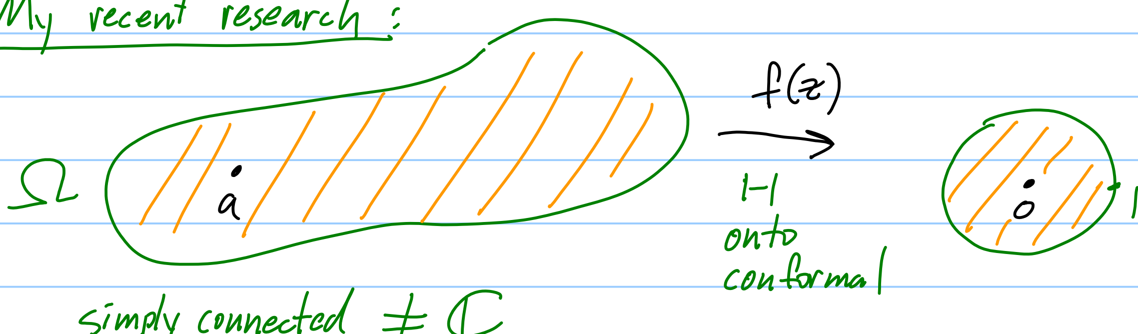
Aha! Every point in box gets hit exactly once.

Details: Choosing $x_1, x_2, x_3$, $A, B$ to line up rectangle.
Done numerically. Elliptic integrals arise.

Fancy cases.




My recent research:




← Poisson integral formula on $D_1(0)$!

Disc: Nice: Averaging property

$$h(0) = \underbrace{\frac{1}{2\pi r}}_{L} \int_0^{2\pi} h(re^{i\theta}) \underbrace{r\,d\theta}_{ds} = \text{Ave with respect to arc length.}$$

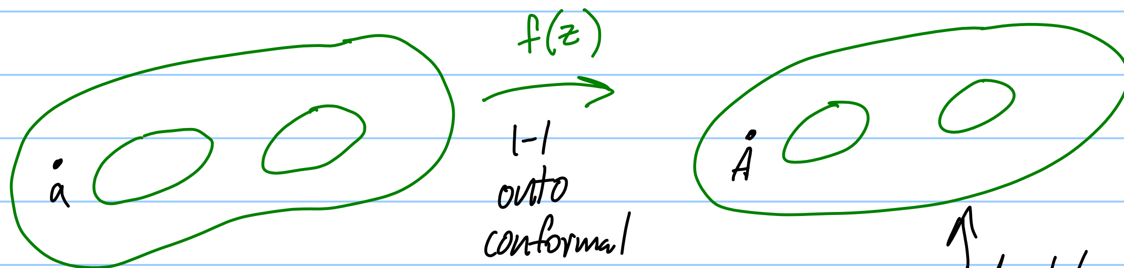
Hmmm. Integrate with respect to $dr$

$$\int_0^1 2\pi r\, h(0)\, dr = \int_0^1 \int_0^{2\pi} h(re^{i\theta})\, d\theta\ r\, dr$$

$$\underbrace{2\pi\, h(0) \left[\tfrac{1}{2} r^2\right]_0^1}_{h(0)\ \pi \cdot 1^2} = \iint_{D_1(0)} h(z)\, dA$$

$$h(0) = \frac{1}{\text{Area}\,(D_1(0))} \iint_{D_1(0)} h\, dA$$

Two averaging properties: With respect to arc length, and area.




double quadrature domain

Nice Poisson formula

$$\iint h\, dA = \sum_{k=1}^{N} c_k\, h^{(k)}(A)$$

$$\int h\, ds = \sum_{k=1}^{N} b_k\, h^{(k)}(A)$$