Lesson 41 Local and Open Mapping Theorems

38, 39 due Wed.

[Final Exam: Tues, 7–9 pm
Dec. 12, LILY 3410

Review material on home page.

Local mapping theorem: $f(z_0) = w_0$, f not constant fcn.

$F(z) = f(z) - w_0$ has zero at w_0. It must be isolated, and have an order $N \geq 1$.

$$f(z) - w_0 = a_N (z-z_0)^N + \cdots = (z-z_0)^N \Big[\underbrace{\overset{\neq 0}{a_N} + a_{N+1}(z-z_0) + \cdots}_{H(z)} \Big]$$

$H(z_0) = a_N \neq 0$

Big idea: Since $H(z_0) \neq 0$, can cook up an N-th root fcn.

• $a_N = H(z_0) \neq 0$

$$h(z) = e^{\frac{1}{N} \log H(z)}$$ is an analytic N-th root of H near z_0.

$$f(z) - w_0 = (z-z_0)^N \underbrace{H(z)}_{= h(z)^N} = \Big[\underbrace{(z-z_0)\, h(z)}_{g(z)} \Big]^N$$

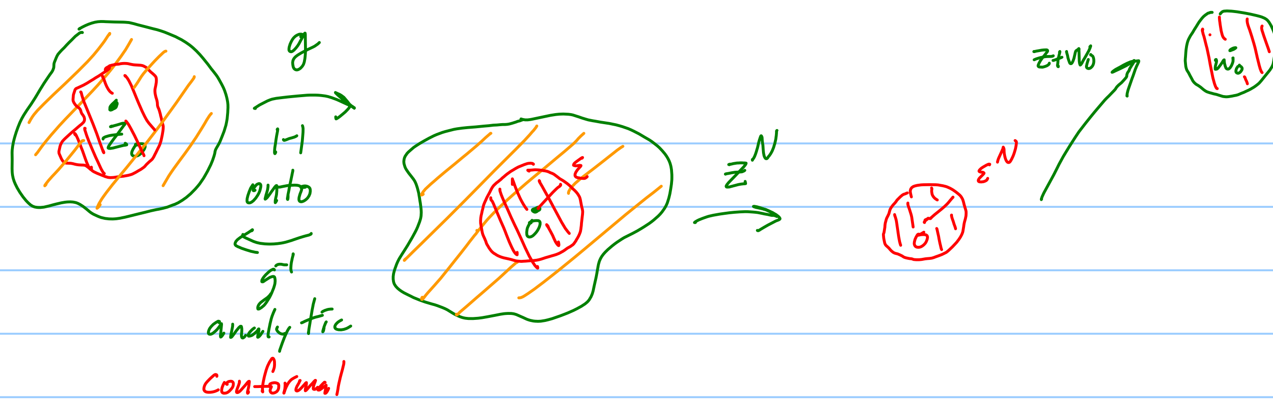
Claim: $g'(z_0) \neq 0$. $\quad h(z_0)^N = H(z_0) \neq 0$. So $h(z_0) \neq 0$.

$$g'(z) = 1 \cdot h(z) + (z-z_0)\, h'(z)$$
$$g'(z_0) = h(z_0) + 0 \neq 0.$$

Inverse fcn theorem yields that g is locally 1-1, onto, invertible, and conformal!

$$f(z) = w_0 + [g(z)]^N$$

Aha! f = (FIRST $g(z)$), then N-th power, translate by w_0.



Cor: Open mapping theorem: A non-constant analytic function maps open sets to open sets.

Cor: If f is analytic and one-to-one on an open set, then f' is non-vanishing!

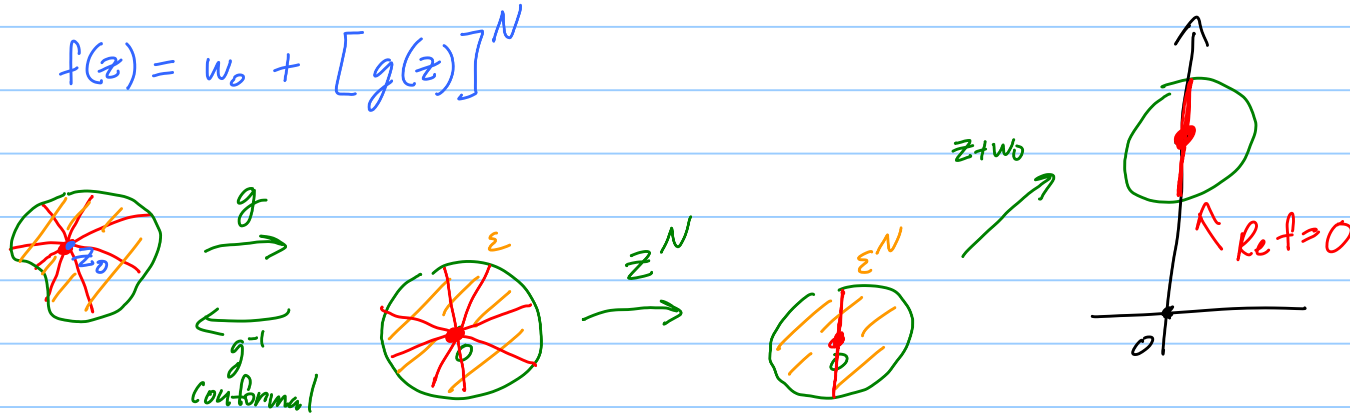
Way false $\mathbb{R}\to\mathbb{R}$: $h(t) = t^3$ $[-1,1] \longrightarrow [-1,1]$ 1-1, onto

h is C^∞-smooth. h^{-1} not diff'ble at $t=0$.

Cor: What does the zero set of a harmonic function look like?

u harmonic. $u(z_0)=0$. Let $f=u+iv$, v harm conj.

$$f(z) = w_0 + [g(z)]^N$$



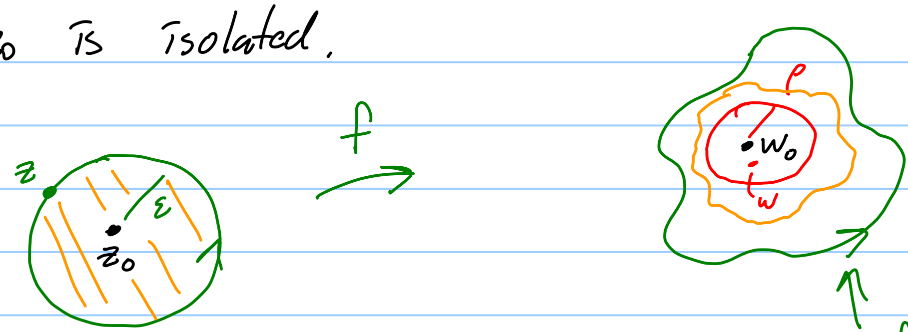
Aha! Zeroes of non-constant harmonic fcns cannot be isolated.

The zero set consists of smooth curves crossing at angles $\frac{2\pi}{N}$ for positive integers N.

Rouché's Theorem $\Rightarrow$ Open Mapping Theorem

Suppose f non-constant, analytic. $f(z_0) = w_0$.

non-const $\Rightarrow$ $f(z) - w_0$ is not $\equiv 0$. Hence zero at z_0 is isolated.



z_0 is the only zero of $f(z) - w_0$ on $\overline{D_\varepsilon(z_0)}$.

$f(C_\varepsilon(z_0))$ misses w_0.

Claim: Every w in $D_\rho(w_0)$ gets hit by f

Rouché: $|H| < |F|$ on $C_\varepsilon(z_0)$, then F and $F+H$ have same # zeroes inside $C_\varepsilon(z_0)$

$F(z) = f(z) - w_0 \leftarrow$ has at least one zero

$F(z) + H(z) = f(z) - w \leftarrow$ want w to get hit!

$H = f - w - F$

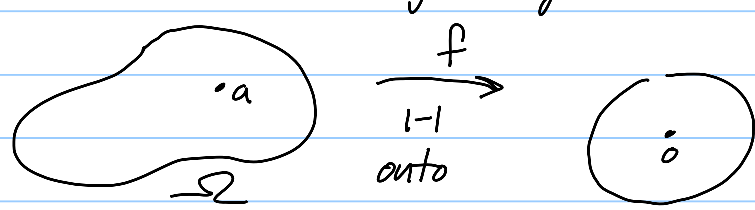
$= f - w - (f - w_0) = w_0 - w$

Hmmm. Do I have $|H| < |F|$ on $C_\varepsilon(z_0)$?

$$\underbrace{|w_0 - w|}_{<\rho} \overset{?}{<} \underbrace{|f(z) - w_0|}_{>\rho} \qquad \text{Yes!}$$

because $\rho < \min\limits_{C_\varepsilon(z_0)} |f(z) - w_0|$

Remark: The Schwarz Lemma is a major ingredient of Riemann Mapping Thm proof.



f is Riemann map

Suppose $g : \Omega \to D_1(0)$ is analytic and $g(a) = 0$.

Today: Know f^{-1} is analytic (because f is 1-1).

Aha! $H = g \circ f^{-1}$ maps $D_1(0) \to D_1(0)$ and $H(0) = 0$.

Schwarz: $|H'(0)| \le 1$.

$$H(z) = g(f^{-1}(z))$$

$$H'(0) = g'(\underbrace{f^{-1}(0)}_{a})\underbrace{(f^{-1})'(0)}_{\frac{1}{f'(a)}}$$

Get $$\left| g'(a) \frac{1}{f'(a)} \right| \le 1$$

$$|g'(a)| \le |f'(a)|$$

If equality holds, see $g = \lambda f$.

Riemann map has biggest possible $|f'(a)|$ at a among all analytic maps into $D_1(0)$!