Review lecture 1

Final Exam Tues Dec 12 7-9 pm
in LILY 3410

1. Find a harmonic conjugate to $u(x, y) = x^3y - xy^3 + 2x + 3y$ on $\mathbb{C}$.

$f = u + iv$

C-R Eqns $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$v_x = -u_y = -(x^3 - 3xy^2 + 3)$ ← $v = \int \cdots \, dx$

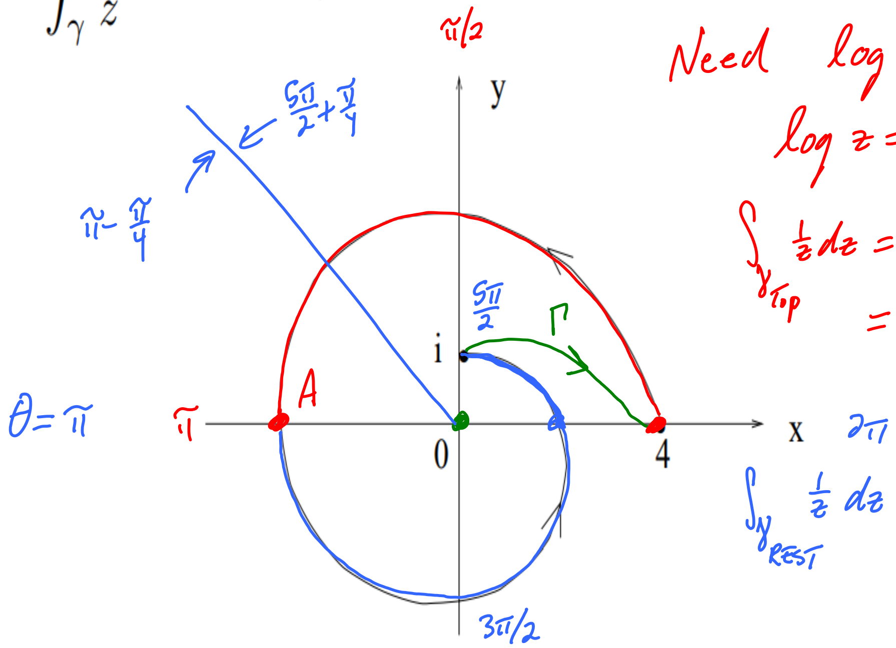
$= \underline{\text{function} + C(y)}$

$v_y = u_x = 3x^2y - y^3 + 2$ ← $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(\text{function}) + C'(y)$

solve for $C'(y)$.

Integrate to get $C(y)$

2. Compute $\displaystyle\int_\gamma \frac{1}{z}\, dz$ where γ is the curve depicted below. Explain your work.



Need $\log z$'s!

$\log z = \ln|z| + i\theta$

$\displaystyle\int_{\gamma_{Top}} \frac{1}{z}\, dz = \log A - \log 4$

$= \underbrace{\ln|A| + i\pi}_{} - (\ln 4)$

$=$

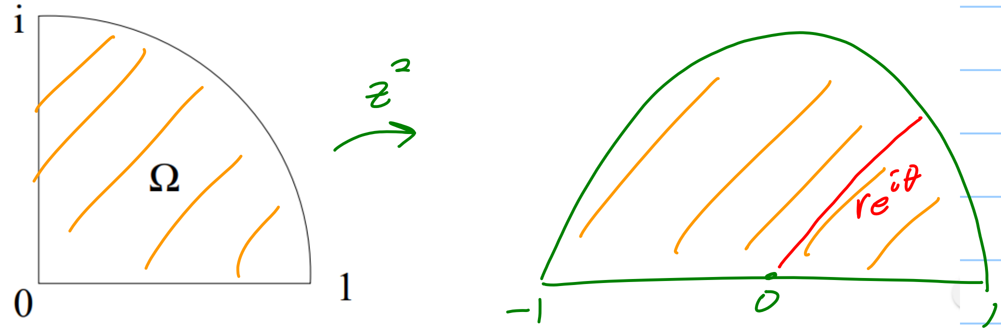
$\displaystyle\int_{\gamma_{REST}} \frac{1}{z}\, dz = \log i - \log A$

$$\int_\gamma \frac{1}{z}\, dz = \log i - \ln 4 = \Big(\underbrace{\ln|i|}_{0} + i\frac{5\pi}{2}\Big) - \ln 4$$

$$\int_{\text{Closed}} \frac{1}{z}\, dz = 2\pi i \cdot 1 \qquad \int_\gamma = \int_{\text{Closed}} - \underbrace{\int_\Gamma \frac{1}{z}\, dz}_{\text{Log } 4 - \text{Log } i}$$

$$\int_{\text{spiral}} \frac{1}{z}\, dz = \ln|\text{END}| - \ln|\text{START}| + i \underset{\text{Spiral}}{\Delta} \arg z$$

3. Let $f_1(z) = z^2$, $f_2(z) = \text{Log } z$, and $f_3(z) = -\frac{i}{\pi}z$. Draw a sequence of three diagrams to illustrate the effect of the mapping $f(z) = f_3(f_2(f_1(z)))$ on the domain Ω below.



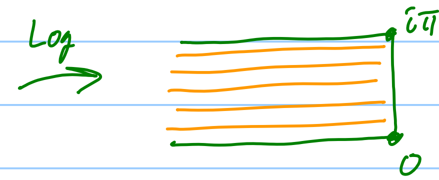
$0 < r < 1$

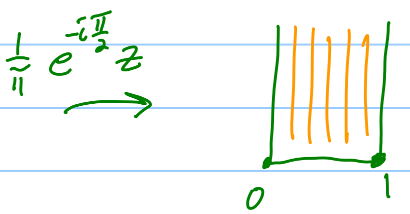
$\text{Log } re^{i\theta} = \text{Ln } r + i\theta \qquad 0 < \theta < \pi$

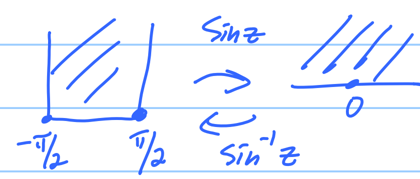
$0 < r\ 1$

$-\infty < \text{Ln } r < 0$

$-i = e^{-i\frac{\pi}{2}}$







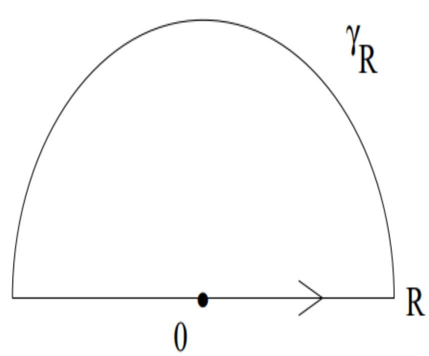
4. Prove that the power series $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(n!)^2}$ converges to an entire function. Compute

$$\int_{\gamma_R} \underbrace{\left(\overset{H'(z)}{\frac{1}{(z-i)^3}} + \overset{G'(z)}{\frac{2}{(z-i)^2}} + \overset{\text{Res}}{\frac{3}{(z-i)}} + \overset{R=\infty \ \text{entire} \quad f(z)}{\sum_{n=0}^{\infty} \frac{(z-i)^n}{(n!)^2}} \right)}_{\text{Laurent Expansion at } z=i} dz$$

Pole of order 3 at i

where γ_R is the contour below with $R > 1$. Explain.

$H(z) = \frac{1}{-3+1}(z-i)^{-3+1}$



$I = 2\pi i \, \text{Res}_i = 2\pi i \cdot 3$

5. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ is a power series about $z = 0$ with a positive radius of convergence. The quotient

$$f(z) = \frac{\sum_{n=0}^{\infty} a_n z^n}{e^z}$$

$$e^z f(z) = \sum_0^\infty a_n z^n$$

has a power series expansion $\sum_{n=0}^{\infty} c_n z^n$.

$$\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(c_0+c_1 z+\cdots\right) =$$

a) What is the general formula for a_n in terms of the c_j's?

b) Compute c_0, c_1, and c_2 in terms of the a_n's.

Left: $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{c_k}{(n-k)!}\right) z^n = \sum_0^\infty a_n z^n$ RIGHT

$$(1\cdot c_0) + (1\cdot c_1 + 1\cdot c_0)z + \left(1\cdot c_2 + 1\cdot c_1 + \tfrac{1}{2!}c_0\right)z^2 + \cdots$$

$c_0 = a_0 \leftarrow$ got c_0 $\qquad c_0 = a_0$

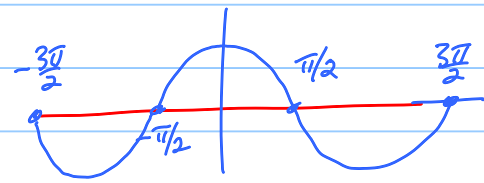
$c_1 + c_0 = a_1 \leftarrow$ get c_1 here $\qquad c_1 = a_1 - c_0 = a_1 - a_0$

$c_2 + c_1 + \frac{1}{2}c_0 = a_2 \leftarrow \quad c_2 = -(a_1 - a_0) - \frac{1}{2}a_0 + a_2$

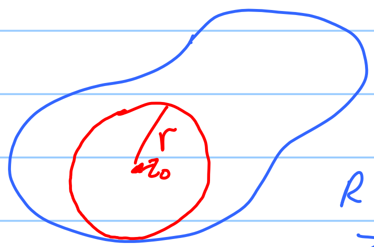
6. The function which is equal to 1 at zero and $\frac{\tan z}{z}$ away from zero has a power series expansion $\sum_{n=0}^{\infty} a_n z^n$ at $z = 0$. What is the radius of convergence of this series? What is a_1?

$\frac{\tan z}{z} = \frac{\tan z - \tan 0}{z - 0} \to \frac{d}{dz} \tan z \Big|_{z=0} = 1$ $\quad z=0$ is removable

$\tan z = \frac{\sin z}{\cos z}$ ← ouch at zeroes of Cos



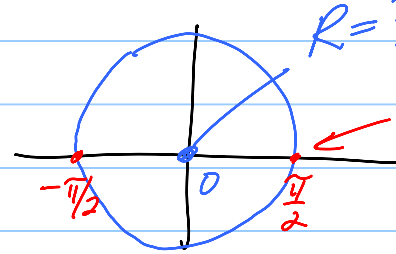
$\frac{\pi}{2} + n\pi$



f analytic on Ω

R of C at z_0 $\geq r$

$R = \frac{\pi}{2}$



← simple poles at $z = \pm\frac{\pi}{2}$

7. Find all the points in the set $\{z : |z| \leq 1\}$ where $\underbrace{z^4 + z}_{f(z)}$ attains its maximum modulus.

Max Princ I: If $|f|$ has a local max in Ω, then $f \equiv$ const.

Max Princ II: If is continuous on $\overline{\Omega}$ ← Ω bounded
and analytic on Ω,
then $|f|$ attains its max on the boundary.

Boundar $= C_1(0)$ $\quad z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

$$|f(z)| = |z^4 + z| = |z(z^3 + z)| = \underbrace{|z|}_{\substack{=1 \\ \text{on } C_1(0)}} |z^3 + 1| = |(e^{i\theta})^3 + 1|$$

$$= |e^{i3\theta} + 1|$$

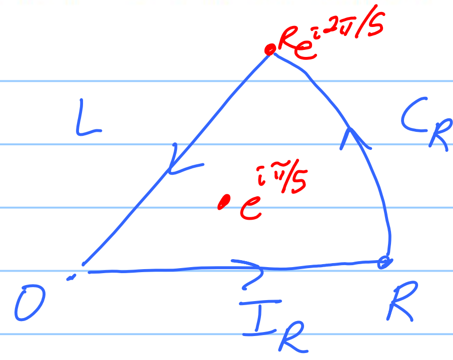
$$= |\cos 3\theta + i \sin 3\theta + 1|$$

$|f(e^{i\theta})|^2 = (\cos 3\theta + 1)^2 + \sin^2 3\theta = 2 + 2\cos 3\theta$ ← easy!

8. Compute

$$\int_0^\infty \frac{1}{x^5+1}\,dx$$

by integrating $f(z) = 1/(z^5+1)$ around the contour that follows the real line from zero to R, then follows the circle Re^{it} from $t = 0$ to $t = 2\pi/5$, and then follows the line $te^{i2\pi/5}$ from $t = R$ back to $t = 0$. Use the Residue Theorem and let $R \to \infty$.



$f(z) = \frac{1}{z^5+1}$

What is $\text{Res}_{e^{i\pi/5}} f(z)$?

$-L: z(t) = t e^{i2\pi/5}$, $0 \le t \le R$. $\quad z'(t) = e^{i2\pi/5}$

$$\int_L f(z)\,dz = -\int_{-L} f(z)\,dz = -\int_0^R \frac{1}{\underbrace{(te^{i2\pi/5})^5}_{t^5} + 1} \cdot \underbrace{e^{i2\pi/5}}_{z'(t)}\,dt$$

9. Convert the integral

$$\int_0^{2\pi} \frac{d\theta}{(2+\sin\theta)^2}$$

into a contour integral of the form $\int_C f(z)\,dz$ where f is a rational function and C is the unit circle parametrized in the standard sense. DO NOT COMPUTE THE VALUE of the integral and don't even bother to use any algebra to clean up the function.

10. Compute the first two coefficients in the Laurent expansion for the function

$$f(z) = \frac{e^z}{z - \sin z}$$

at $z_0 = 0$ (corresponding to the powers z^{-3} and z^{-2}).