Review lecture 2

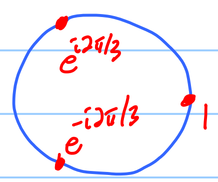
Final Exam Tues, Dec 12, 7-9 pm in LILY 3410

Office Hours: Fri, Mon, Tues 2-3 pm

7. Find all the points in the set $\{z : |z| \le 1\}$ where $z^4 + z$ attains its maximum modulus.

$f(z) = z^4 + z = z(z^3+1)$ $|f(e^{i\theta})|^2 = 2 + 2\cos 3\theta$ ← max at $\theta = 0, \pm \frac{2\pi}{3}$

z^4+z not constant. So max modulus can't be inside disc.

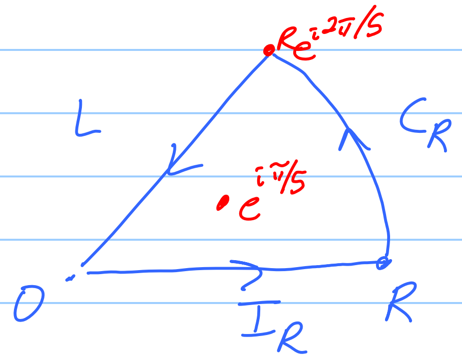


Max modulus of 2 at these 3 pts.

8. Compute

$$\int_0^\infty \frac{1}{x^5+1}\,dx$$

by integrating $f(z) = 1/(z^5+1)$ around the contour that follows the real line from zero to R, then follows the circle Re^{it} from $t = 0$ to $t = 2\pi/5$, and then follows the line $te^{i2\pi/5}$ from $t = R$ back to $t = 0$. Use the Residue Theorem and let $R \to \infty$.



$f(z) = \frac{1}{z^5+1}$

What is $\operatorname{Res}_{e^{i\pi/5}} f(z)$?

$-L: z(t) = te^{i2\pi/5}, \; 0 \le t \le R.$ $z'(t) = e^{i2\pi/5}$

$$\int_L f(z)\,dz = -\int_{-L} f(z)\,dz = -\int_0^R \frac{1}{\underbrace{(te^{i2\pi/5})^5}_{t^5}+1} \cdot \underbrace{e^{i2\pi/5}}_{z'(t)}\,dt$$

$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$ if z_0 is a simple zero of g

$\operatorname{Res}_{e^{i\pi/5}} \frac{1}{z^5+1} = \frac{1}{5(e^{i\pi/5})^4}$ ← not zero!

$\operatorname{Res}_{z_0} \frac{F}{z-z_0} = F(z_0)$

$$\left| \int_{C_R} \frac{1}{z^5+1}\, dz \right| \le \left(\operatorname*{Max}_{C_R} \frac{1}{|z^5+1|} \right)\left(\frac{2\pi}{5} R\right) \le \frac{1}{R^5-1} \cdot \frac{2\pi}{5} R \quad \text{if } R>1.$$

$$P(z) = z^5 + 4z^4 + 3z^3 + 2z^2 + z + 1$$

Basic poly estimate: $a|z|^5 \le |P(z)| \le A|z|^5$ for $|z| > R_0$.

$$\left| \frac{1}{P(z)} \right| \le \frac{1}{a} |z|^{-5} \quad \text{if } |z| > R_0.$$

Thm: $\deg P \le \deg Q - 2$, Q no zeroes on $\mathbb{R}$. $\displaystyle\int_{-\infty}^{\infty} \frac{P}{Q}\, dx = 2\pi i \sum_{UHP} \operatorname{Res} \frac{P}{Q}$

Thm: $\deg P \le \deg Q - 1$, Q no zeroes on $\mathbb{R}$, $\displaystyle\int_{-\infty}^{\infty} \frac{P}{Q} e^{ix}\, dx = 2\pi i \sum_{UHP} \operatorname{Res} \frac{P}{Q} e^{iz}$

9. Convert the integral

$$I = \int_0^{2\pi} \frac{d\theta}{(2+\sin\theta)^2} \overset{?}{=} \int_{C_1(0)} \sim\, dz$$

into a contour integral of the form $\int_C f(z)\, dz$ where f is a rational function and C is the unit circle parametrized in the standard sense. DO NOT COMPUTE THE VALUE of the integral and don't even bother to use any algebra to clean up the function.

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$$

$C_1(0)$: $z(\theta) = e^{i\theta}$

$z'(\theta) = i e^{i\theta}$

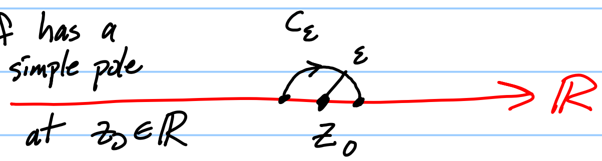
$$I = \int_0^{2\pi} \frac{1}{\left(2 + \frac{z(\theta) + \frac{1}{z(\theta)}}{2i}\right)^2} \frac{i e^{i\theta}\, d\theta}{i e^{i\theta}}$$

$dz = z'(\theta)\, d\theta$

$= i\, z(\theta)$

$$= \int_{C_1(0)} \frac{1}{\left(2 + \frac{z + \frac{1}{z}}{2i}\right)^2} \cdot \frac{1}{iz}\, dz = 2\pi i \sum_{\substack{\text{inside} \\ C_1(0)}} \operatorname{Res}$$

Fact: f has a simple pole at $z_0 \in \mathbb{R}$



$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\, dz = -\frac{1}{2}\left(2\pi i \operatorname{Res}_{z_0} f(z)\right)$$

10. Compute the first two coefficients in the Laurent expansion for the function

$$f(z) = \frac{e^z}{z - \sin z}$$

at $z_0 = 0$ (corresponding to the powers z^{-3} and z^{-2}).

$$f(z) = \frac{e^z}{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{e^z}{\frac{z^3}{3!} - \frac{z^5}{5!} + \cdots} = \frac{e^z}{z^3\Big[\underbrace{\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots}_{H(z)}\Big]}$$

Read off: $H(0) = \frac{1}{3!}$, $H'(0) = 0$, $H''(0) = -\frac{2!}{5!}$

$$= \frac{1}{z^3}\Big[\underbrace{\frac{e^z}{H(z)}}_{A_0 + A_1 z + A_2 z^2 + \cdots}\Big]$$

$$= \frac{A_0}{z^3} + \frac{A_1}{z^2} + \frac{A_2}{z} + \cdots$$

$$A_0 = \frac{e^0}{H(0)} = \frac{1}{1/3!} = 3! \qquad A_1 = \frac{1}{1!}\frac{d}{dz}\left[\frac{e^z}{H(z)}\right]\Bigg|_{z=0} = \frac{e^z H(z) - e^z H'(z)}{H(z)^2}\Bigg|_{z=0}$$

$$= \frac{e^0 \frac{1}{3!} - e^0 \cdot 0}{(1/3!)^2} = 3!$$

$$\mathrm{Res}_0 = A_2 = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{e^z}{H(z)}\right]\Bigg|_{z=0} \quad \text{ouch!}$$

Only 3 types of isolated singularities: $f(z)$ analytic on $D_\varepsilon(z_0) - \{z_0\}$

1) z_0 is removable: $\lim_{z \to z_0} f(z)$ exists. ← no negative terms in Laurent Exp

→ Fact: If $|f|$ is bounded on $D_\rho(z_0) - \{z_0\}$ for some $0<\rho\varepsilon$, z_0 is removable,

2) z_0 is a pole: $\lim_{z \to z_0} f(z) = \infty$ $\leftarrow$ finitely many neg terms

3) z_0 is essential: $\underbrace{f(D_\rho(z_0) - \{z_0\})}_{\substack{\text{Picard's} \; = \mathbb{C} \\ \text{or } \mathbb{C} - \{a\}}}$ is dense in $\mathbb{C}$ no matter how small ρ is $0 < \rho < \varepsilon$, ∞ many neg terms.

$\llcorner$ Riemann removable singularity theorem.

Problem: Show that a pole of f cannot be a pole of e^f.

z_0 is a pole of $f(z)$. $f(z) = (z - z_0)^{-N} F(z)$ $F(z_0) \neq 0$.

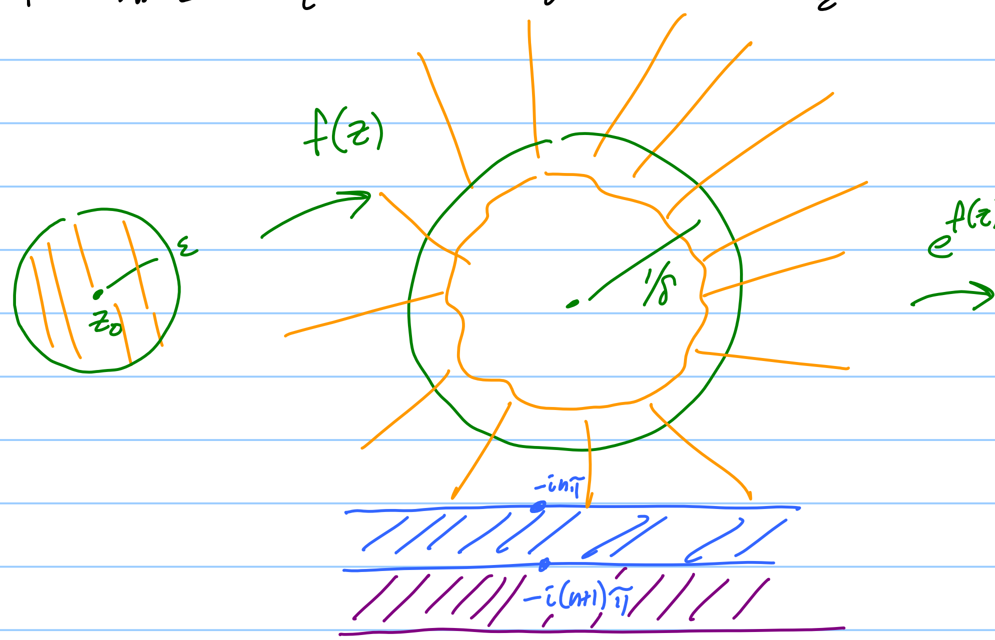
So $\frac{1}{f(z)}$ has a removable sing at z_0.

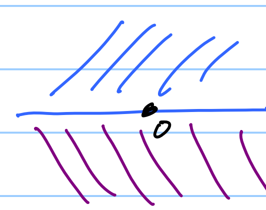
In fact, a zero of order N.

f not constant. Open mapping thm $\frac{1}{f}$ maps open sets to open sets.

A disc $D_\delta(0)$ gets hit by $\frac{1}{f}$ on $D_\varepsilon(z_0)$.

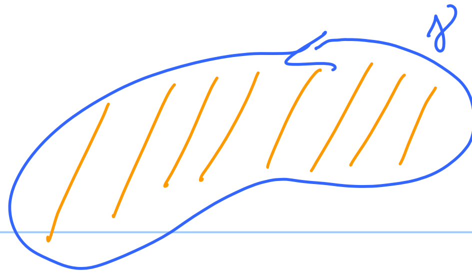
So f hits $\{z : |z| > \frac{1}{\delta}\}$ on $D_\varepsilon(z_0) - \{z_0\}$.





e^f has an essential sing where f has pole.

Rouché's Thm:



f, h analytic inside and on simple closed γ.

$|h| < |f|$ on γ.

Then f and $f+h$ have same number of zeroes inside γ.

Prob: How many zeroes does $\underset{\uparrow \, 2^4=16}{z^4} - \underset{\uparrow \, \underline{\underline{32}}}{4z^3} + \underset{\uparrow \, 12}{6z} - \underset{\uparrow \, 3}{3}$ have inside $C_2(0)$?

big one

$f(z) = -4z^3 \leftarrow$ 3 zeroes inside $C_2(0)$ counted with mult.

$h(z) = z^4 + 6z - 3 \leftarrow$ the rest of the poly.

$$|h(z)| \le |z^4| + |6z| + |3| \underset{\substack{\uparrow \\ \text{on } C_2(0)}}{\le} \underbrace{2^4 + 12 + 3}_{\substack{16+12+3 \\ =31}} < 32 = |-4z^3| = |f(z)|$$

on $C_2(0)$

Rouché's: f and $\underbrace{f+h}_{\text{our poly!}}$ both have 3 zeroes in $C_2(0)$