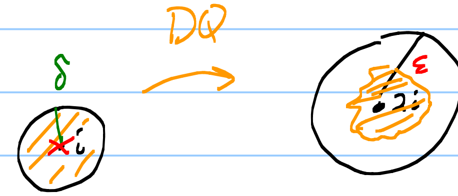
Lesson 5 The Cauchy-Riemann equations

Lessons 4,5,6 due Wed. Read Chap 2.

Thurs office hour moved to today.

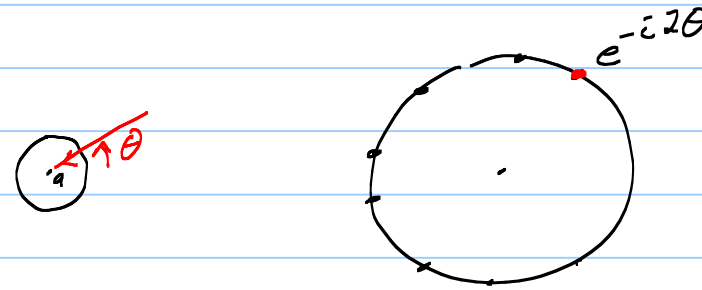
EX: DQ for z^2 at $a=i$.

$DQ \to 2i$ as $z \to i$



EX: DQ for $\bar{z}$

$z = a + re^{i\theta}$ $\quad DQ = e^{-2i\theta}$



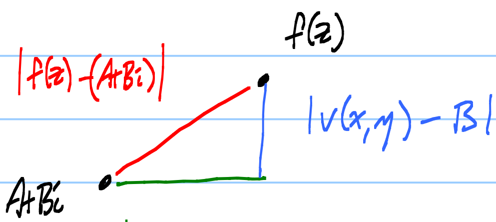
Lemma: Write $z = x+iy$ $\quad z_0 = x_0 + iy_0$

$$f(x+iy) = u(x,y) + iv(x,y)$$

Then

$$\lim_{z\to z_0} f(z) = A+Bi \iff \begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = A \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = B \end{cases}$$

Pf:

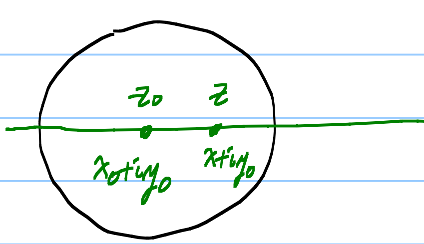


$(\Rightarrow)$ leg $\triangle \le$ hyp

$(\Leftarrow)$ Pythagorean

Cauchy-Riemann Eqns: $\quad DQ = \dfrac{f(z)-f(z_0)}{z-z_0}$ $\quad$ Write $u_0 = u(x_0,y_0)$

Horizontal slide



$$\frac{f(z)-f(z_0)}{z-z_0} =$$

$$\frac{[u(x,y_0)+i\,v(x,y_0)]-[u_0 \quad +i\,v_0 \quad]}{(x+iy_0)-(x_0+iy_0)}$$

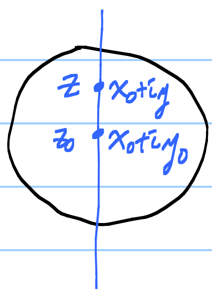
$$\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

f $\mathbb{C}$-diff'ble at z_0 $\Rightarrow$ limit exists as $x \to x_0$

Aha! Lemma $\Rightarrow$ $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exist at (x_0, y_0)

and $\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$

Vertical slide



$$DQ = \frac{(u(x_0,y)-u(x_0,y_0)) + i\,(v(x_0,y)-v(x_0,y_0))}{\underbrace{(x_0+iy)-(x_0+iy_0)}_{i(y-y_0)}}$$

$\frac{1}{i} = -i$

$$= \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0} + i\left(-\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0}\right)$$

See $\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ exist at (x_0,y_0) and

$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$ Box 2

$\|$

$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$ Box 1

Conclude:

Theorem: If $f = u + iv$ is complex diff'ble at z_0, then all first partial derivatives of u and v exist at (x_0, y_0)

and $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ at (x_0, y_0).

Ex: $f(z) = \bar{z}$ $\qquad f(x+iy) = x - iy$

$u(x,y) = x \qquad v(x,y) = -y$

Oops! C-R Eqn #1 fails! $\bar{z}$ can't be comp. diff'ble!

Ex: $f(z) = e^z = e^x e^{iy}$ $\qquad f(x+iy) = (e^x \cos y) + i(e^x \sin y)$

Hmmm, C-R Eqns are satisfied!

Good news!, Reverse implication holds.

Theorem: $f(x+iy) = u(x,y) + i\,v(x,y)$

If $\underbrace{u \text{ and } v \text{ are } C^1\text{-smooth}}_{\text{(annoying to assume)}}$ $\leftarrow$ $\left\{ \begin{array}{l} u, v \text{ continuous,} \\ \text{first partials exist} \\ \text{and are continuous} \end{array} \right.$

and satisfy the C-R Eqns, then f is complex diff'ble.

Consequence: e^z is complex diff'ble on $\mathbb{C}$!

i.e., e^z is $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire fcn}}$.

More detailed C-R Eqn fact: Toss in two red boxes:

$$f'(z) = \begin{cases} \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial v}{\partial y} - i\,\dfrac{\partial u}{\partial y} \end{cases} \quad \longleftarrow \quad E'(z) = E(z) \ !$$

Prob: $E(z) = e^z$. What is $E'(z)$? $\longleftarrow$ C-R Eqns show E' exists

$= (e^x \cos y) + i\,(e^x \sin y)$

Big fact: $E'(z) = E(z)$

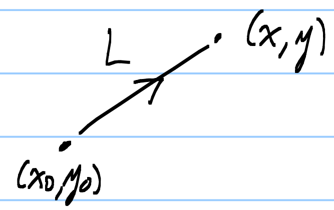
Note: Tossing in $E(0) = 1$ and $E' = E$ uniquely determines $E(z)$.

Calculus lemma: If $u(x,y)$ is $\mathcal{C}^1$-smooth, then

$$u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) + R(x,y)$$

where
$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Why:



$L: (x_0,y_0) + t\,(x-x_0,\, y-y_0) \quad 0 \le t \le 1$

$$u(x,y) - u(x_0,y_0) = \int_0^1 \underbrace{\frac{\partial}{\partial t}\, u\big(\overbrace{x_0 + t(x-x_0),\ y_0 + t(y-y_0)}^{eee}\big)}_{u_x(eee)\cdot(x-x_0) + u_y(eee)\cdot(y-y_0)}\, dt$$

Hmmm.
$$\int_0^1 u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)\, dt = \text{first order taylor term}$$

Subtract:

$$\underbrace{u - u_0 - u_x^0(x-x_0) - u_y^0(y-y_0)}_{R} =$$

$$\int_0^1 \underbrace{\left[u_x(\ldots) - u_x^0\right]}_{|\;| \le \frac{\varepsilon}{2} \text{ when } |z-z_0|<\delta}(x-x_0) + \underbrace{\left[v_y(\ldots) - v_y^0\right]}_{|\;| \le \frac{\varepsilon}{2}}(y-y_0)\; dt$$

$$\frac{|R|}{|z-z_0|} \le \frac{\varepsilon}{2}\frac{|x-x_0|}{|z-z_0|} + \frac{\varepsilon}{2}\frac{|y-y_0|}{|z-z_0|} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$