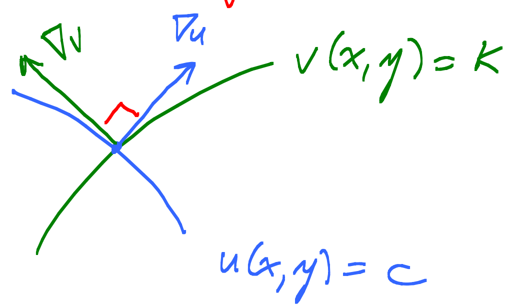
Lesson 7 Harmonic conjugates

Great fact: Level sets of harmonic conjugates are $\perp$

Why: u harmonic. $f = u + i\underset{\substack{= \\ \uparrow}}{v}$ is analytic

v harm conj for u

C-R Eqns: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$



∇u is a normal vector to $u=c$.

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \underbrace{\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)}_{\left(-\frac{\partial u}{\partial y},\, \frac{\partial u}{\partial x}\right)} = 0$$

Fact: Steady state solutions to the heat equ are harmonic.

$$\underbrace{\frac{\partial u}{\partial t}}_{\to 0} = c \Delta u \qquad u = \text{temp in a metal plate.}$$

$\to 0$ as temp stabalizes, leaving $\Delta u = 0$

EX: $u(x,y) = x^3 - 3xy^2 + y$ is harmonic, $\Delta u \equiv 0$ ✓

Look for harm conj v so that $u + iv$ is analytic.

Need $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

(A) $v_x = -u_y = -\frac{\partial}{\partial y}(x^3 - 3xy^2 + y) = 6xy - 1$

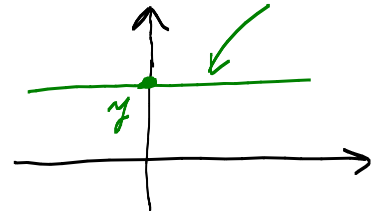
(B) $v_y = u_x = 3x^2 - 3y^2$

Step 1: Use (A): $v = \int 6xy - 1 \, dx = 3x^2 y - x + \underbrace{C(y)}_{\text{arb fcn of } y}$

<u>Why</u>: Note: $\frac{\partial}{\partial x}(3x^2y - x) = \underline{6xy - 1}$

$$\frac{\partial v}{\partial x} = \text{Same thing}$$

So $\frac{\partial}{\partial x}\Big(\underbrace{v - [3x^2y - x]}_{\text{const on horiz lines! It is a fcn of } y!}\Big) \equiv 0$



Call it $C(y)$

(A) gave us $\boxed{v = \underline{3x^2y - x + C(y)}}$

<u>Step 2</u>: Use (B): $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$$\frac{\partial}{\partial y}\Big(\underbrace{3x^2y - x + C(y)}_{v}\Big) \overset{\text{want}}{=} 3x^2 - 3y^2$$

$$3x^2 - 0 + C'(y) = 3x^2 - 3y^2$$

$$\boxed{C'(y) = -3y^2}$$

So $C(y) = \int -3y^2\, dy = -y^3 + K$

<u>Step 3</u>: $v = 3x^2y - x + (-y^3 + K)$

$f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic!

$x = \frac{z+\bar{z}}{2}$
$y = \frac{z-\bar{z}}{2i}$

plug in. All $\bar{z}$'s cancel! Get

$$f(z) = z^3 - i(z - K)$$

<u>Easy lemma</u>: If f is analytic on a domain Ω and $f' \equiv 0$ on Ω, then f is constant on Ω.

Why: Case $\Omega = D_r(a)$. $f = u + iv$

Red box #1: $f' = u_x + i v_x$

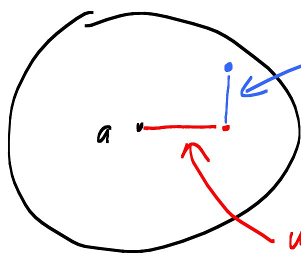
Red box #2: $f' = v_y - i u_y$

Aha! If $f' \equiv 0$, then $\nabla u \equiv 0$ and $\nabla v \equiv 0$

Obvious that u, v are const?

Freshman calculus: $h(t)$ is continuous on $[a, b]$ and $h'(t)$ exists and is equal to zero on (a, b). Then $h(t) \equiv c$, a constant. Pf: MVT

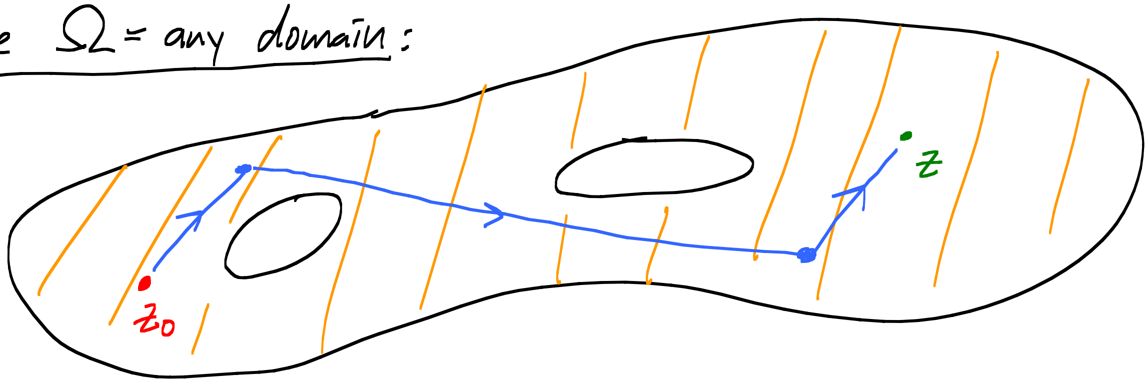
also const along vert because $\frac{\partial}{\partial y} = 0$



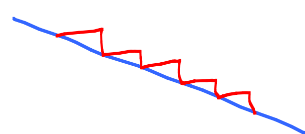
u, v const along horiz because $\frac{\partial}{\partial x} = 0$

See u, v are constant on disc!

Case Ω = any domain:



Aha! Can zig and zag along blue lines



Big fact: If f is analytic, then so is f'!

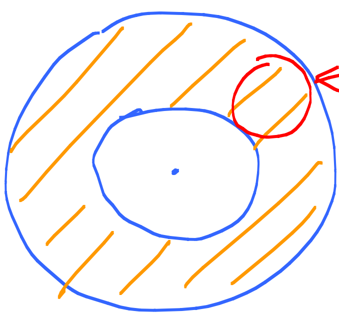
So f is infinitely complex diff'ble.

Two facts: 1) C-R Eqns
2) f' exists $\Rightarrow$ f' continuous $\Big\}$ $\Rightarrow$ u, v are infinitly continuously diff'ble in x and y!

Assume Big Fact from now on. So real and imag parts are harmonic fcns (because we don't have to assume u, v C^2-smooth).

Important example: $u = \underbrace{Ln(x^2+y^2)^{1/2}}_{Ln|z|}$ is harmonic on

$$\Omega = \{z : 1 < |z| < 2\}$$



get harm conj v here

Get $v(x,y) = \underbrace{Arg(z)}_{\theta} + K$

Ouch! Can't go around the hole and make v hook up.

Fact: u harmonic on a domain Ω with no holes, then u has a harmonic conjugate on Ω.
Maybe not, if Ω has holes!