# Lesson 10 Complex powers, heat problems

Liouville: "Elementary functions": $e^z$, $\log z$, trig fcns, hyp trig fcns, and inverses, algebraic fcns.

Thm: $\int e^{x^2}dx$ cannot be expressed as an elem fcn!

No problemo! $\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$ = Area

Chain rule for analytic fcns: $f, g$ analytic and $h(z) = f(g(z))$. $[g: \Omega_1 \to \Omega_2,\ f: \Omega_2 \to \mathbb{C}]$

Then $h$ is analytic $[$on $\Omega_1]$ and

$$h'(z) = f'(g(z))\,g'(z).$$

Notation: $\zeta = f(w)$ where $w = g(z)$.

$$\frac{d\zeta}{dz} = \frac{d\zeta}{dw}\cdot\frac{dw}{dz}$$

Why: $f$ diff'ble: $DQ = \boxed{\dfrac{f(w)-f(A)}{w-A} = f'(A) + E(w)}$

where $E(w) \to 0$ as $w \to A$. Define $E(A) = 0$ to make $E$ continuous at $A$.

Let $A = g(a)$, $w = f(z)$. Solve box for $f(w) - f(A)$:

$$DQ = \frac{f(g(z)) - f(g(a))}{z-a} = f'(g(a))\cdot\frac{(g(z)-g(a))}{z-a} + E(g(z))\frac{(g(z)-g(a))}{z-a}$$

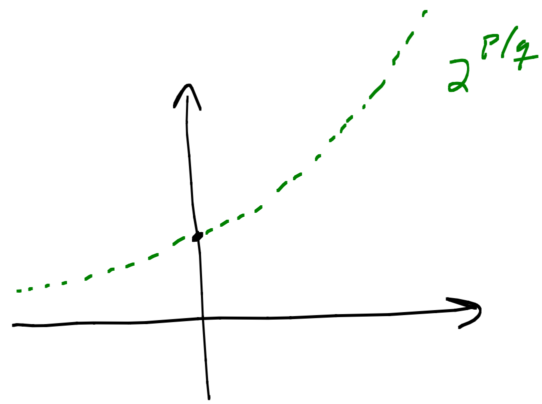
$$h'(a) = f'(g(a)) \cdot g'(a) + 0 \cdot g'(a)$$

Why green 0? $g$ diff'ble $\Rightarrow$ $g$ cont. So, as $z \to a$, $g(z) \to g(a) = A$. So $E(g(z)) \to E(A) = 0$.

Real powers: $\pi^{\sqrt{2}} = ?$

$2^p = ?$ $\quad 2^n = \underbrace{2 \cdot 2 \cdots 2}_{n\text{-times}}$

$$2^{p/q} = \sqrt[q]{(2^p)}$$




$$2^\pi = \lim\ 2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \ldots$$

$$\boxed{2^\pi = e^{\ln 2^\pi} = e^{\pi \ln 2}} \leftarrow \text{Def}^n \text{ of } 2^\pi!$$

Complex version: "$z^p$" $= e^{p \log z}$ $\leftarrow$ lots of values!

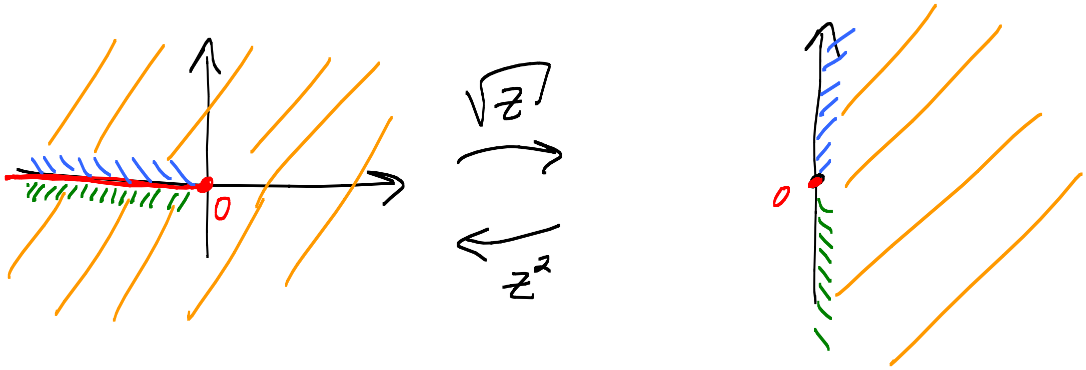
$z^p = e^{p \operatorname{Log} z}$ is the "Principal branch" of $z^p$.

$\sqrt{z} = e^{\frac{1}{2}\operatorname{Log} z}$ is the princ branch of sq root.

$$\sqrt{re^{i\theta}} = e^{\frac{1}{2}\operatorname{Log} re^{i\theta}} = e^{\frac{1}{2}[\ln r + i\theta]}$$

where $-\pi < \theta \le \pi$

$$= e^{\frac{1}{2}\ln r}\, e^{i\theta/2} = \sqrt{r}\, e^{i\theta/2}$$




How many square roots of w: = z such that $z^2 = w$.

$$z = e^{\frac{1}{2}\log w} = e^{\frac{1}{2}\left[\mathrm{Ln}|w| + i \arg w\right]}$$

$$= e^{\frac{1}{2}\mathrm{Ln}|w|} \quad e^{i\frac{1}{2}\left[\mathrm{Arg}\, w + i 2\pi n\right]}$$

$$= |w|^{1/2} \quad e^{i\frac{1}{2}\mathrm{Arg}\, w} \quad \underbrace{e^{i\pi n}}_{= \pm 1 \,!}$$

So $\quad z^{1/2} = \pm\sqrt{z}$

princ sq rt.

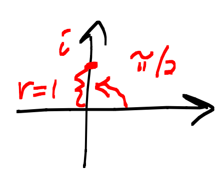
Easy facts: $\quad z^n$ only has one value

$$z^n = e^{n\log z} = e^{n\mathrm{Ln}|z|} \quad e^{in\,\mathrm{Arg}\, z} \quad \underbrace{e^{in\cdot m 2\pi}}_{1}$$

$z^{p/q} = \left(z^p\right)^{1/q}$ has exactly $q$ values.

EX: $\quad i^i = ?\quad$ Princ value: $\quad i^i = e^{i\,\mathrm{Log}\, i}$

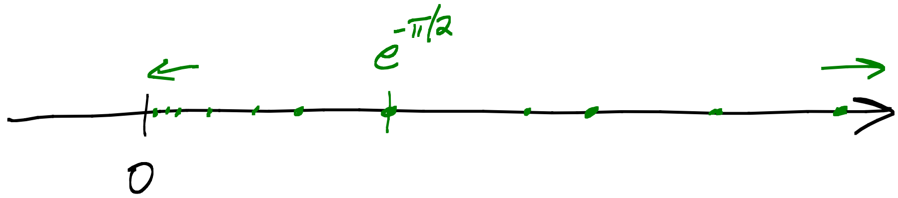
$$= e^{i\left[\underbrace{\mathrm{Ln}|i|}_{0} + i\frac{\pi}{2}\right]} = e^{-\pi/2}$$



All possible values: $e^{i\left[\underbrace{\mathrm{Ln}|i|}_{0} + i\left(\frac{\pi}{2}+2\pi n\right)\right]}$

$$i^i = \left\{ e^{-\frac{\pi}{2}-2\pi n} \;:\; n \in \mathbb{Z} \right\}$$

All real and positive. Some $\to 0$ and some $\to +\infty$!



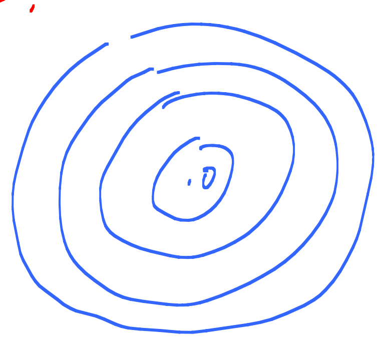

## 3.4 Washers, wedges, walls.

$$\mathrm{Log}\, re^{i\theta} = \mathrm{Ln}\, r + i\theta \qquad -\pi < \theta < \pi \quad \text{is analytic}$$

$$= \underbrace{\mathrm{Ln}\sqrt{x^2+y^2}}_{} + i\,\underbrace{\mathrm{ArcTan}\,\frac{y}{x}}_{}$$

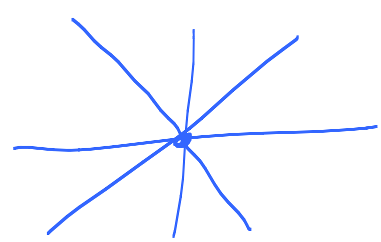
harmonic!

$\mathrm{Ln}\, r$: Level sets




circles centered at 0

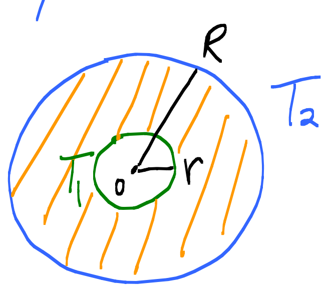
$\theta$: Level sets



radial lines

Solve steady state heat by "inspection".

Prob.:




Guess temp fcn $u$

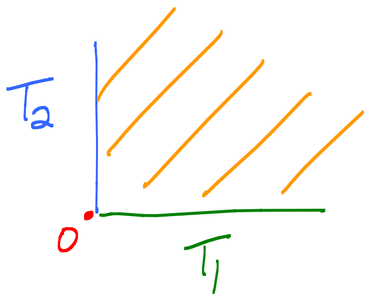
$u = \mathrm{Ln}|z|$ Hmmm.

$\frac{T_2}{\ln R} \cdot \ln|z|$ good on outer circ.

Inner circ: $\frac{T_2}{\ln R} \cdot \ln r$

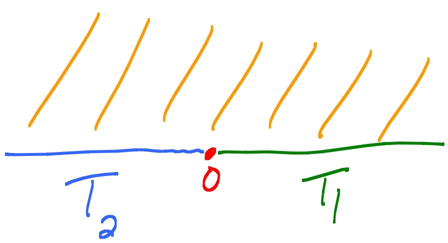
Aha! $u = A + B \ln|z|$ ← two conditions determine two constants.

Prob:



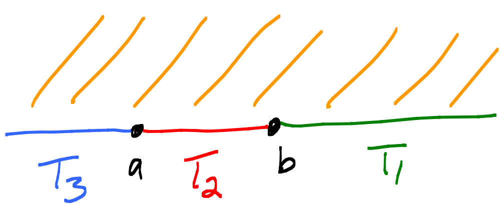

$u = A + B\,\mathrm{Arg}\, z$

Prob:



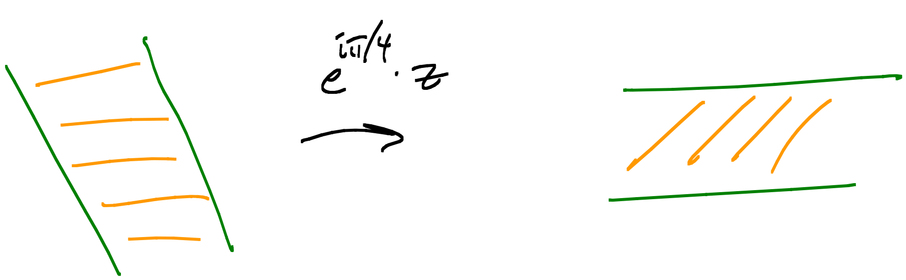

$u = A + B\,\mathrm{Arg}\, z$

Bigger problem:




Hmmm: $u = A + B \cdot \mathrm{Arg}(z-a) + C \cdot \mathrm{Arg}(z-b)$

Hwk:




$y = \mathrm{Im}\, z$