Lesson 11 Complex integration $\int_\gamma f \ dz$

9,10,11 due Wed

Remark: Inverse Fcn Theorem + Chain rule $\Longrightarrow$

Inverse Fcn Formula: $F = f^{-1}$

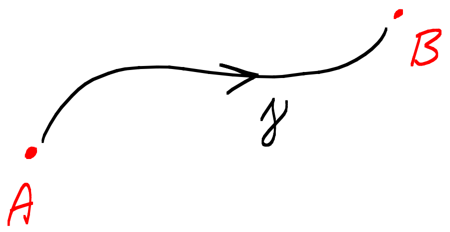
$$\text{Since} \quad F(f(z)) = z$$

$$F'(f(z))\, f'(z) = 1$$

$$\text{So} \quad F'(f(z)) = \frac{1}{f'(z)} \quad \text{or} \quad F'(w) = \frac{1}{f'(F(w))}$$

$$w = f(z) \qquad \frac{dz}{dw} = \frac{1}{\left(\frac{dw}{dz}\right)}$$

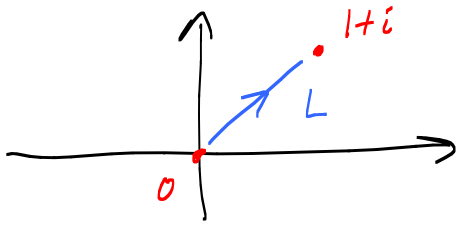
Curves in $\mathbb{C} \approx \mathbb{R}^2$:



$$\gamma: \ (x(t), y(t)), \quad a \le t \le b$$

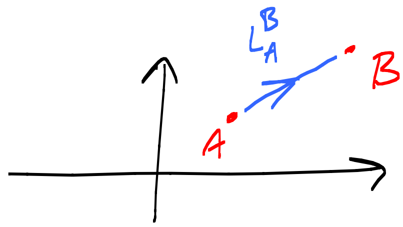
$$\text{Better:} \quad z(t) = x(t) + i y(t), \quad a \le t \le b.$$

EX:



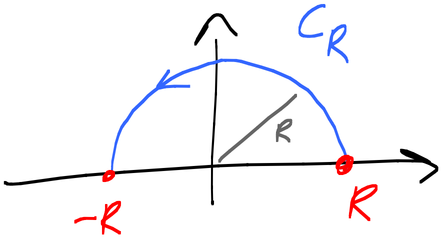
$$L: \ z(t) = t(1+i)$$
$$0 \le t \le 1$$

EX:



$$z(t) = A + t(B-A)$$
$$0 \le t \le 1$$

EX:



$$C_R: \ z(t) = R e^{it}, \quad 0 \le t \le \pi$$

Two kinds of functions:

1) $f : \Omega \to \mathbb{C}$ analytic

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

2) $z(t)$ $\qquad z : [a,b] \to \mathbb{C} \qquad z(t) = x(t) + i y(t)$

Define
$$z'(t_0) = \lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0}$$
$$= \lim_{t \to t_0} \left[\frac{x(t) - x(t_0)}{t - t_0} + i \, \frac{y(t) - y(t_0)}{t - t_0} \right]$$
$$= x'(t_0) + i y'(t_0)$$

Two kinds of integrals:

1) Easy kind:

$$\int_a^b z(t)\, dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n) \Delta t$$
$$= \lim_{\Delta t \to 0} \left[\sum_{n=1}^{N} x(t_n) \Delta t + i \sum_{n=1}^{N} y(t_n) \Delta t \right]$$
$$\overset{\text{Def}^n}{=} \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt$$

Assuming $x(t), y(t)$ cont.

Fundamental Theorem of Calculus #1:

$$\underbrace{\int_a^b z'(t)\,dt}_{\int_a^b x'(t)\,dt + i\int_a^b y'(t)\,dt} = z(b) - z(a)$$

✓

Assuming $x(t)$, $y(t)$ continuously diff'ble.

Basic inequality: $$\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$$

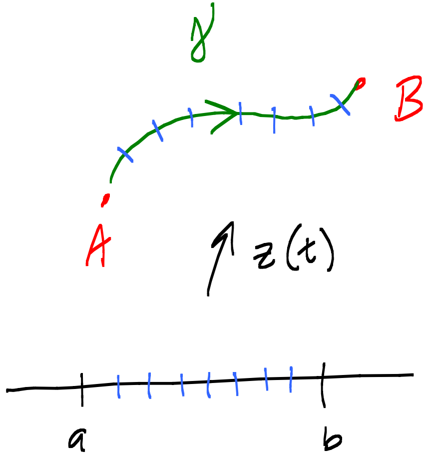
Why: $$\left|\sum_{n=1}^{N} z(t_n)\Delta t\right| \le \sum_{n=1}^{N} |z(t_n)|\,\Delta t$$

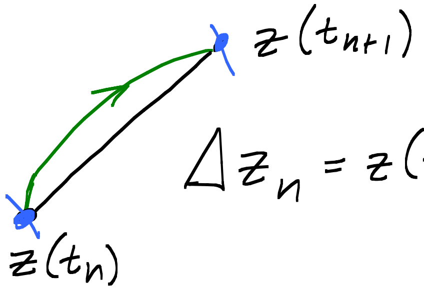
Let $\Delta t \to 0$. ✓

Baby lemma: $$\int_a^b \underset{\uparrow\, k\in\mathbb{C}}{k}\, z(t)\,dt = k\int_a^b z(t)\,dt$$

Why: Write $k = A + Bi$, $z = x + iy$. Write out, multiply out. ✓

Integration of $f(z)$ along a curve in $\mathbb{C}$:





$$\Delta z_n = z(t_{n+1}) - z(t_n)$$

$$\int_\gamma f\,dz = \lim_{\Delta t\to 0} \sum_{n=1}^{N} f(z(t_n))\,\Delta z_n$$

$$\underset{\underset{\text{claim}}{\uparrow}}{=} \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{\text{Think } = dz}$$

Why:

$$\frac{z(t)-z(t_n)}{\Delta t} = z'(t_n) + E_n(t) \leftarrow \text{small! if } \Delta t \text{ small}$$

$$\Delta z_n = z'(t_n)\,\Delta t + \underbrace{E_n(t)\,\Delta t}_{\text{super small}}$$

$$\Bigg|\underbrace{\sum f(z(t_n))\,z'(t_n)\,\Delta t}_{\to \int_a^b f(z(t))\,z'(t)\,dt} - \sum f(z(t_n))\,\Delta z_n\Bigg| \le$$

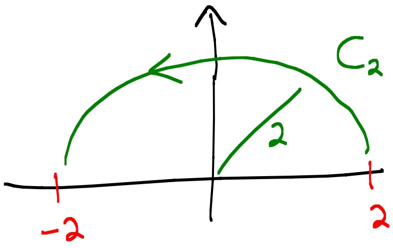
$$\Bigg|\sum f(z(t_n))\,E_n(t_{n+1})\,\Delta t\Bigg|$$

$$\le \underbrace{\Big(\max_\gamma |f|\Big)}_{M}\cdot \varepsilon \underbrace{\sum_1^N \Delta t}_{= b-a}$$

Def^n:

$$\int_\gamma f\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz} \qquad \frac{dz}{dt} = z'(t)$$

Ex:



$$C_2:\quad z(t) = 2e^{it},\ 0 \le t \le \pi$$

$$z'(t) = 2ie^{it}$$

$$\int_{C_2} e^{z}\, dz = \int_0^{\pi} e^{z(t)}\, z'(t)\, dt$$

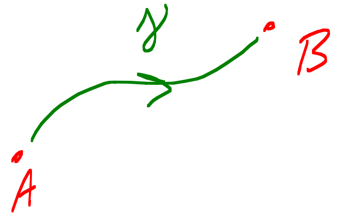
$$= \int_0^{\pi} e^{2e^{it}} \cdot 2i\, e^{it}\, dt$$

$$= 2i\int_0^{\pi} e^{2\cos t + i\, 2\sin t} \cdot e^{it}\, dt$$

$$= 2i\int_0^{2\pi} e^{2\cos t} \cdot e^{i(2\sin t + t)}\, dt \leftarrow \text{Uhg!}$$

$$= \int (\text{Real part})\, dt + i \int (\text{Im part})\, dt$$

Fund. Thm Calculus #2 :



f analytic on open set containing γ

$$\int_\gamma f'\, dz = f(B) - f(A)$$

Easy to show using

Chain Rule #2: $$\frac{d}{dt} f(z(t)) = \underset{\substack{\uparrow \\ \text{complex} \\ \text{deriv}}}{f'}(z(t))\ \underset{\substack{\uparrow \\ \text{today's} \\ \text{new deriv}}}{z'}(t)$$

Write $f(x+iy) = u(x,y) + i\, v(x,y)$

$z(t) = x(t) + i\, y(t)$

Then $f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$ and

$$\frac{d}{dt} f(z(t)) = \left(u_x \dot{x} + \overset{v_x}{u_y}\, \dot{y}\right) + i \left(v_x \dot{x} + \overset{-u_x}{v_y}\, \dot{y}\right) \qquad \text{C-R Eqns}$$

$$= \underbrace{\left[u_x + i v_x\right]}_{f'} \cdot \underbrace{(\dot{x} + i \dot{y})}_{z'} \qquad ✓$$

EX: $\frac{d}{dt} e^{i3t} = \frac{d}{dt} E(i3t) = E'(i3t)\, \frac{d}{dt}(i3t)$

$$= E(i3t) \cdot (i3)$$

$$= i3e^{i3t} \qquad ✓$$

Pf of Fund Thm Calc #2: $\int_\gamma f'\, dz = \int_a^b f'(z(t))\, \underbrace{z'(t)\, dt}_{dz}$

$$= \int_a^b \frac{d}{dt}\left[f(z(t))\right] dt \underset{\substack{\uparrow \\ \text{Fund Thm} \\ \text{Calc \#1}}}{=} f(\underbrace{z(b)}_{B}) - f(\underbrace{z(a)}_{A})$$

Back to Uhg! $\int_{C_2} e^z\, dz = \int_{C_2} \frac{d}{dz}(e^z)\, dz = e^2 - e^{-2}$

Basic estimate #2: $\left| \int_\gamma f\, dz \right| \le \left(\underset{\gamma}{\text{Max}} |f| \right) \cdot \text{Length}(\gamma)$