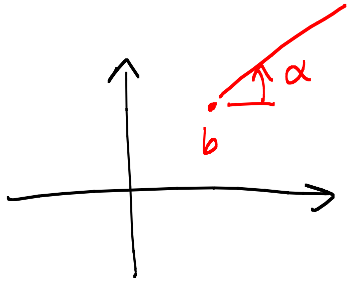
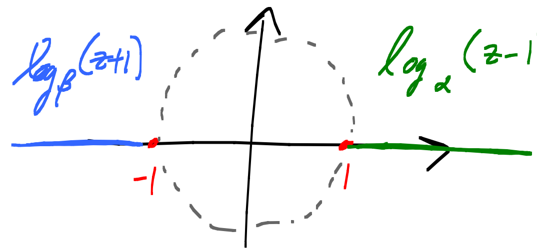
# Lesson 13 Cauchy's Theorem

12, 13, 14 due next Wed

Exam 1 in class Wed, Oct. 3




$\log_\alpha(z-b)$



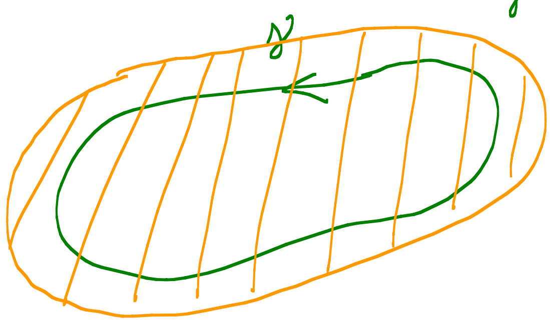

$$(z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2}$$

$$= \underbrace{e^{\frac{1}{2}\log_\alpha(z-1)} \cdot e^{\frac{1}{2}\log_\beta(z+1)}}$$

Square this: get $e^{\log_\alpha(z-1)} \cdot e^{\log_\beta(z+1)} = (z-1)(z+1) = z^2-1$

Read 4.4b instead of 4.4a.

**Cauchy's Theorem:** Suppose $f$ is analytic "inside and on" a simple closed curve $\gamma$. Then $\int_\gamma f\,dz = 0$.



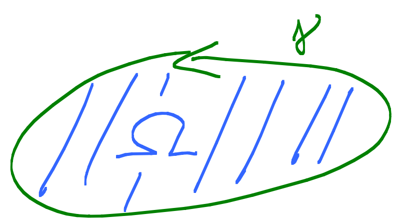

[hatched box] is a bigger open set containing $tr(\gamma)$ and the inside of $\gamma$.

**Jordan curve Theorem:** A simple closed curve separates $\mathbb{C}$ into

$$\{\text{inside of } \gamma\} \cup tr(\gamma) \cup \{\text{outside of } \gamma\}.$$

bounded open connected sets unbounded

**Green's Theorem:**




$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Stokes': $\int_\gamma \omega = \iint_\Omega d\omega$ $\qquad \omega = P\,dx + Q\,dy$

$$d\omega = d(P\,dx + Q\,dy)$$
$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)\wedge dy$$

Use $\quad dx\wedge dx = 0$, $\quad dy \wedge dy = 0$ $\qquad dy\wedge dx = -\underbrace{dx\wedge dy}_{=+d(\text{Area})}$

Pf of Cauchy's: (C-R Eqns + Green's)

$\gamma: z(t) = x(t) + iy(t)$, $a \le t \le b$

$f(x+iy) = u + iv$

$$\int_\gamma f\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$
$$= \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right](\dot{x}(t) + i\dot{y}(t))\,dt$$
$$= \int_a^b (u\dot{x} \overset{i^2=-1}{-} v\dot{y})\,dt + i\int_a^b (v\dot{x} + u\dot{y})\,dt$$
$$= \left(\int_\gamma u\,dx - v\,dy\right) + i\left(\int_\gamma v\,dx + u\,dy\right)$$

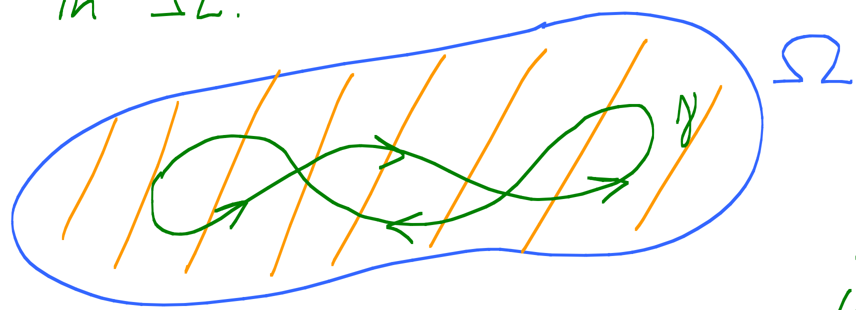
$$\boxed{\int_a^b u(x(t),y(t))\underbrace{\frac{dx}{dt}dt}_{dx} = \int_\gamma u\,dx}$$

$$\overset{\text{Green's}}{=} \iint_\Omega \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\equiv 0} dA + i\iint_\Omega \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\equiv 0} dA = 0$$

by C-R Eqns!

Cauchy's Theorem #2: Suppose $\Omega$ is a domain with no holes, i.e., $\Omega$ is a simply connected domain, and $f$ is analytic on $\Omega$. Then $\int_\gamma f\,dz = 0$ for every closed curve $\gamma$ in $\Omega$.

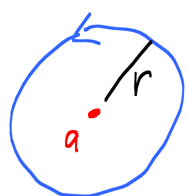
Why:




Cut $\gamma$ into $n$-simple closed loops. Use previous version on each loop.

Cor: Using big fact from last time: Analytic fcns on a domain without holes have analytic antiderivatives.

Important calculation:




$C_r(a)$ : $z(t) = a + re^{it}$

$0 \le t \le 2\pi$

$z'(t) = 0 + ire^{it}$

$$\boxed{\int_{C_r(a)} \frac{1}{z-a}\,dz = 2\pi i}$$

$$\| $$

$$\int_0^{2\pi} \frac{1}{[a+re^{it}]-a}\underbrace{(ire^{it}\,dt)}_{z'(t)dt\ =dz} = \int_0^{2\pi} i\,dt = 2\pi i \checkmark$$

Consequently, version #2 of Cauchy Thm fails for $\underbrace{\mathbb{C}-\{a\}}_{\Omega}$.

$\frac{1}{z-a}$ is analytic on $\Omega$, but $\int_{C_r(a)} \frac{1}{z-a}\,dz \neq 0$.

EX: $\ln|z|$ is harmonic on $\underbrace{\mathbb{C}-\{0\}}_{\Omega}$, but has no harmonic conjugate on $\Omega$.

Suppose $v$ is a harmonic conj. on $\Omega$.


Hmmm. $\operatorname{Log} z = \operatorname{Ln}|z| + i \underbrace{\operatorname{Arg} z}$

Harmonic on $\mathbb{C} - (-\infty, 0]$

But $v$ is harmonic on too!

Recall: Harmonic conjugates are <u>unique</u> up to $+C$.

$$V = \operatorname{Arg} z + C \quad \text{on} \quad \mathbb{C} - (-\infty, 0].$$

Ouch! $v$ is not continuous across cut! No such $v$.

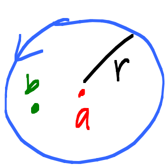
<u>Another consequence</u>: $\frac{1}{z-a}$ cannot have an analytic antideriv on $\mathbb{C} - \{a\}$. Because if $\frac{1}{z-a} = F'(z)$, then

$$2\pi i = \int_{C_r(a)} \frac{1}{z-a}\, dz = \int_{C_r(a)} F'(z)\, dz = F(\text{END}) - F(\text{START}) = 0$$

same

<u>Important fact</u>: $\displaystyle \int_{C_r(a)} \frac{1}{z-b}\, dz = \begin{cases} 2\pi i & \text{if } b \text{ is inside } C_r(a) \\ 0 & \text{if } b \text{ is outside} \end{cases}$

by Cauchy's Thm!




<u>Why</u>: $\displaystyle F(w) = \int_{C_r(a)} \frac{1}{z-w}\, dz$

Show $\displaystyle F'(w) = \int_{C_r(a)} \frac{1}{(z-w)^2}\, dz = \int_{C_r(a)} \frac{d}{dz}\left[ -\frac{1}{z-w} \right] dz = 0$

So $F(w) \equiv 2\pi i$.