Lesson 14 The Cauchy Integral Formula

Last time: $C_r(a):\ z(t) = a + re^{it},\ 0 \le t \le 2\pi$

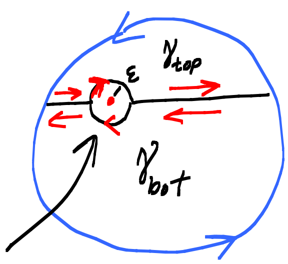
$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{[a+re^{it}]-a}\left(ire^{it}\,dt\right) = \int_0^{2\pi} i\,dt = 2\pi i$$

$$\int_{C_r(a)} \frac{1}{z-b}\,dz = \begin{cases} 2\pi i & \text{if } b \text{ inside } C_r(a) \\ 0 & \text{if } b \text{ outside} \end{cases}$$

by Cauchy's Thm.

One way to see:

$\bullet = b$



$-C_\varepsilon(b)$

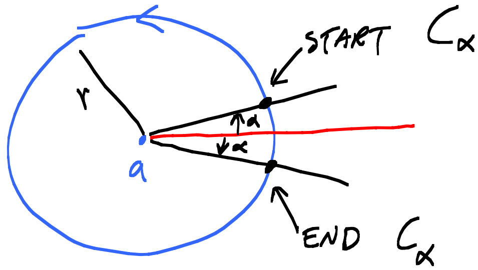
clockwise small circle

Cauchy's Thm $\Longrightarrow$

$$0 = \int_{\gamma_{top}} \frac{1}{z-b}\,dz + \int_{\gamma_{bot}} \frac{1}{z-b}\,dz$$

$$= \int_{C_r(a)} \frac{1}{z-b}\,dz + \underbrace{\int_{-C_\varepsilon(b)} \frac{1}{z-b}\,dz}_{-2\pi i}$$

Another way:



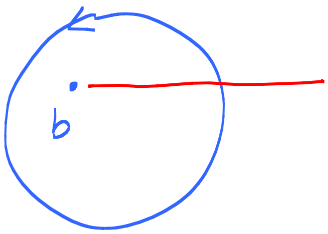
$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \lim_{\alpha\to 0} \underbrace{\int_{C_\alpha} \frac{1}{z-a}\,dz}$$

$$= \int_{C_\alpha} \frac{d}{dz}\left[\log_0 (z-a)\right] dz$$

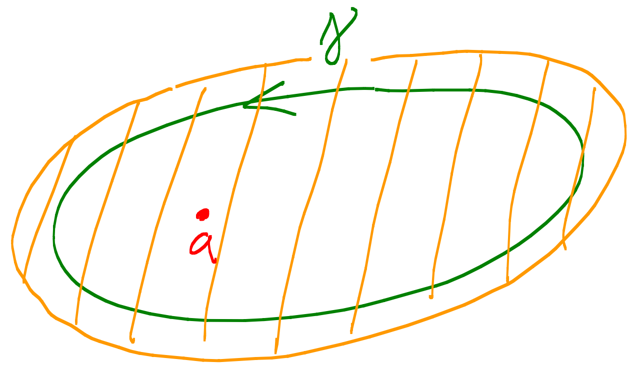
$$= \lim_{\alpha\to 0} \log_0(z-a)\Big|_{a+re^{i\alpha}}^{a+re^{-i\alpha}}$$

$$= \lim_{\alpha\to 0} \left\{ \left[\operatorname{Ln} r + i(2\pi-\alpha)\right] - \left[\operatorname{Ln} r + i\alpha\right] \right\}$$

$$= \lim_{\alpha \to 0} \left(2\pi i - 2i\alpha \right) = 2\pi i \quad \checkmark$$



Same proof works for b inside $C_r(a)$.



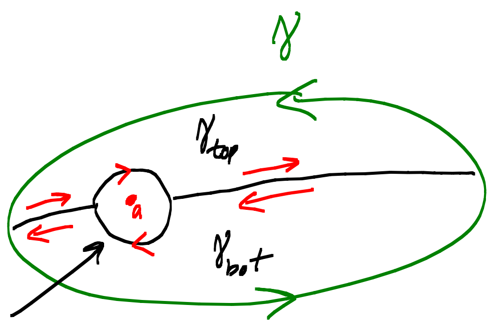
Cauchy Integral Formula:

Assume f is analytic inside and on γ and a is inside γ. Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw$$

counterclockwise

Pf:



$-C_\varepsilon(a)$

Cauchy's Thm $\Longrightarrow$

$$0 + 0 = \left(\int_{\gamma_{top}} + \int_{\gamma_{bot}} \right) \frac{1}{z-a}\, dz$$

$$= \int_\gamma \frac{f(w)}{w-a}\, dw + \underbrace{\int_{-C_\varepsilon(a)} \frac{f(w)}{w-a}\, dw}$$

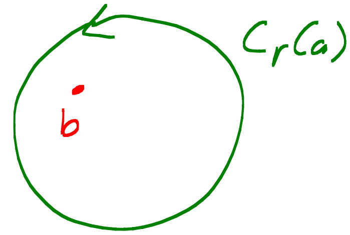
$$= - \int_{C_\varepsilon(a)} \frac{f(w)}{w-a}\, dw$$

$$= - \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{[a+\varepsilon e^{it}] - a} \left(i\varepsilon e^{it}\, dt \right)$$

$$= -i \int_0^{2\pi} \underbrace{f(a+\varepsilon e^{it})}_{\to f(a) \text{ as } \varepsilon \to 0}\, dt \to -i\, f(a)\, 2\pi$$

Let $\varepsilon \to 0$. Get C.I.F.

Important case: $\gamma = C_r(a)$

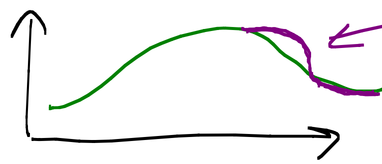


$$f(b) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(z)}{z-b}\, dz$$

Whoa! Values of f inside circle completely determined by values on the circle. Think: Analytic fcns are "rigid." They can't be deformed like continuous fcns can.

like sol'ns to heat probs!

Way false $\mathbb{R} \to \mathbb{R}$:



poke graph and keep it diff'ble $\mathbb{R} \to \mathbb{R}$

Fantastic Fact: Can differentiate under the $\int$ in the C.I.F.! So f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' anal. $\Rightarrow$ ... and u and v are C^∞-smooth fcns!

Why:
$$DQ = \frac{f(z)-f(a)}{z-a} = \frac{1}{2\pi i} \cdot \frac{1}{z-a} \cdot \int_C \underbrace{\left(\frac{f(w)}{w-z} - \frac{f(w)}{w-a} \right)}_{f(w)\frac{(w-a)-(w-z)}{(w-z)(w-a)}} dw$$

$(w-a)-(w-z) = z-a$

pull out the $z-a$ [a constant in the $\int$].

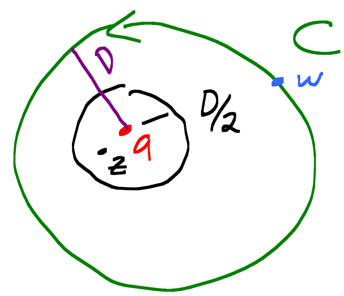
$$= \frac{1}{2\pi i} \int_C f(w) \frac{1}{(w-z)(w-a)} dw \xrightarrow[z \to a]{\text{Guess}} \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^2} dw$$

Check:
$$\left| DQ - \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^2} dw \right| =$$

$$\left| \frac{1}{2\pi i}\int_C f(w)\left[\underbrace{\frac{1}{(w-z)(w-a)} - \frac{1}{(w-a)^2}}_{\frac{z-a}{(w-a)^2(w-z)}}\right]dw\right|$$

$$= \underbrace{|z-a|}_{\substack{\to 0 \\ \text{as } z\to a!}} \cdot \frac{1}{2\pi i} \underbrace{\left|\int_C \frac{f(w)}{(w-a)^2(w-z)}\,dw\right|}_{\text{Just need to bound this.}}$$

$$|f(w)| \le \underbrace{\max_C |f|}_{M}$$



Let $D = \operatorname{dist}(a, C)$.

Assume $|z-a| < \frac{D}{2}$. Then $|w-a| \ge D$.

$|w-z| > D/2$

So $$\left|\frac{1}{(w-a)^2(w-z)}\right| \le \frac{1}{D^2 \cdot \frac{D}{2}}$$

$$|DQ - \text{Lim}| \le \frac{1}{2\pi i}\,\underline{\underline{|z-a|}} \cdot M \cdot \frac{1}{D^2\cdot\frac{D}{2}} \cdot \text{Length}(C) \to 0$$

as $z \to a$.

Got $$\boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^2}\,dw}$$ $\leftarrow$ Can diff under $\int$!

Can repeat:

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}}\, dw}$$

Remark: Condition $\int_\gamma f\, dz = 0$ for closed curves γ

$\iff$ f has an analytic antiderivative

last time

$\iff$ $\int_\Gamma f\, dz$ is path independent