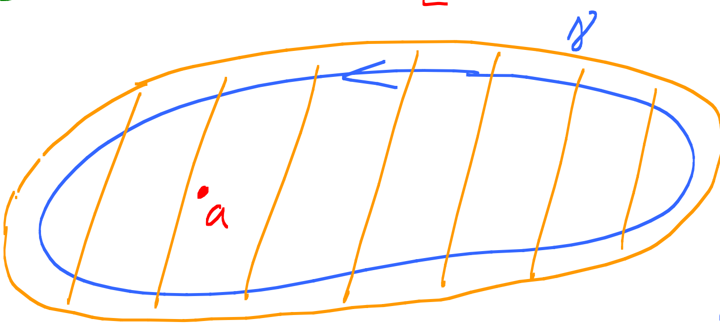
Lesson 15 Liouville's Theorem and the Fundamental Thm of Algebra

12, 13, 14 due Wed., Exam 1 in class: Wed, Oct. 3

Remark: All analytic fcns on a domain with no holes have analytic antiderivatives. [$\frac{1}{z}$ has no antiderivative on $\mathbb{C} - \{0\}$]



γ counterclockwise simple closed curve. f is analytic inside and on γ.

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz \leftarrow \text{C.I. Formula}$$

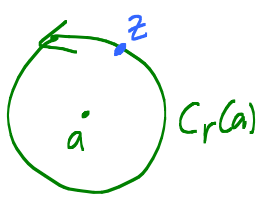
$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2}\, dz \leftarrow \text{Higher order CIF}$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

The Cauchy Estimates: $\gamma = C_r(a)$

$$|f^{(n)}(a)| = \frac{n!}{2\pi} \left| \int_{C_r(a)} \frac{f(z)}{(z-a)^{n+1}}\, dz \right| \le \frac{n!}{2\pi} \underbrace{\left(\max_\gamma |f| \right)}_{M} \frac{1}{r^{n+1}} \underbrace{\text{Length}(C_r(a))}_{2\pi r}$$



$|z-a| = r$ on $C_r(a)$

$$\boxed{|f^{(n)}(a)| \le \frac{n!\, M}{r^n}}$$

Way false $\mathbb{R} \to \mathbb{R}$. $\sin(At)$ can wiggle fast even though $|\sin(At)| \le 1$. [Make A big]

Liouville's Theorem: A $\underbrace{\text{bounded}}_{|f(z)| \le M}$ $\underbrace{\text{entire function}}_{\text{analytic on } \mathbb{C}}$ must be $\underbrace{\text{constant}}_{f(z) \equiv c}$.

Why: Pick $a \in \mathbb{C}$. Assume f is bd entire. $|f(z)| \le M$ for all z.

Cauchy estimate, $n=1$: $\quad |f'(a)| \le \dfrac{M}{r}$ ← can let $r \to \infty$!

Let $r \to \infty$. See $f'(a) = 0$. But a was arbitrary!

So $f'(z) \equiv 0$ and f is constant!

Way false $\mathbb{R} \to \mathbb{R}$. Ex. $\sin t$

Basic polynomial estimate: $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$

where $a_N \ne 0$ and $N \ge 1$. There are real constants $0 < A < B$ and a radius $R_0 > 0$ such that

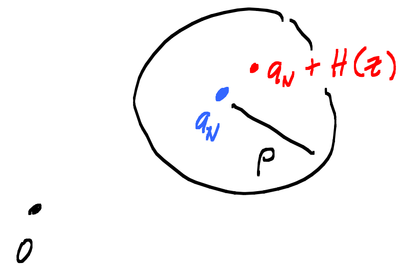
$$A|z|^N < |P(z)| < B|z|^N \quad \text{when } |z| > R_0.$$

Think: $|P(z)|$ blows up like $|z|^N$ as $|z| \to \infty$.

Consequence: Zeroes of P fall inside $C_{R_0}(0)$.

Why: $$P(z) = z^N \Big[\underbrace{a_N}_{\ne 0} + \underbrace{a_{N-1} \cdot \frac{1}{z} + \cdots + a_1 \cdot \frac{1}{z^{N-1}} + a_0 \frac{1}{z^N}}_{H(z)} \Big]$$

Note that $H(z) \to 0$ as $z \to \infty$.



$$\left.\begin{aligned} A &= |a_N| - \rho \\ B &= |a_N| + \rho \end{aligned}\right\} \quad \text{Take } \rho < |a_N|.$$

There is a R_0 such that $|H(z)| < \rho$ if $|z| > R_0$.

$$P(z) = z^N \left[a_N + H(z) \right]$$

$$|P(z)| = |z|^N \underbrace{\left| a_N + H(z) \right|}_{A \le \,\sim\, \le B \text{ if } |z| > R_0.}$$

✓

Remark: Taking ρ small shows $A < |a_N| < B$

as close to $|a_N|$ as desired.

Fundamental Theorem of Algebra: $P(z)$, $N \geq 1$, has a root in $\mathbb{C}$! Consequently we can factor polys in $\mathbb{C}$.

Why: Suppose $P(z)$, $N \geq 1$, has no root in $\mathbb{C}$.

Then $f(z) = 1/P(z)$ is an entire fcn.

Basic poly est shows $|P(z)| \to \infty$ as $z \to \infty$. [Left side.]

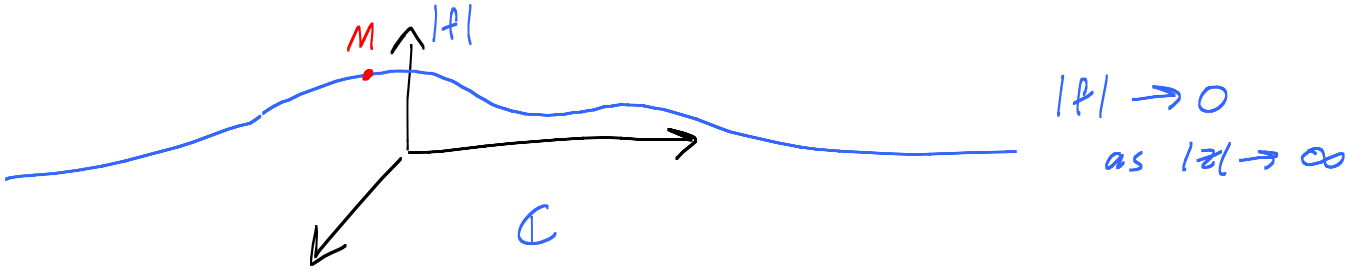
So $1/P(z) \to 0$ as $z \to \infty$.

f is a bounded entire fcn.

Liouville's $\Longrightarrow$ f const. So $P \equiv$ const. ↯

(↯ because $a_N = \dfrac{P^N(0)}{N!}$ is not zero.)

So P must have a root.



Know $|f| \to 0$ as $|z| \to \infty$. So, taking $\varepsilon = 1$, there is a radius R such that $|f(z)| < 1$ if $|z| > R$.

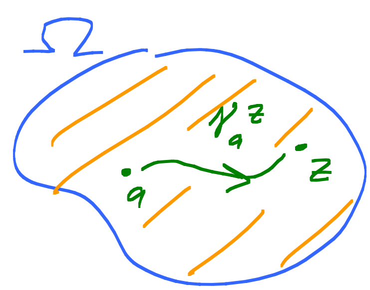
Next, $|f|$ is <u>continuous</u> on $\underbrace{\{z : |z| \leq R\}}_{\text{closed and bounded}}$, so $|f|$ <u>attains</u> its max M there.

Finally, $\text{Max}\{1, M\}$ is a bound for $|f|$ on $\mathbb{C}$.

<u>Fascinating Fact:</u> <u>Morera's Theorem:</u> Given a <u>continuous</u> $f : \Omega \to \mathbb{C}$ on a domain Ω. If $\int_\gamma f\, dz = 0$ for all closed γ in $\mathbb{C}$, then f must be <u>analytic</u>!

We must study $\int_\gamma f\, dz$.

<u>Why</u>: Hmmm. Define
$$F(z) = \int_{\gamma_a^z} f(w)\, dw$$



Morera condition $\Rightarrow$ F <u>well</u> defined.

Next. We showed $F'(z) = f(z)$. So F is analytic.

Aha! F analytic $\Rightarrow$ F' analytic. But $F' = f$. So f is <u>analytic</u>.

Consequence: Uniform limits of polynomials are analytic!