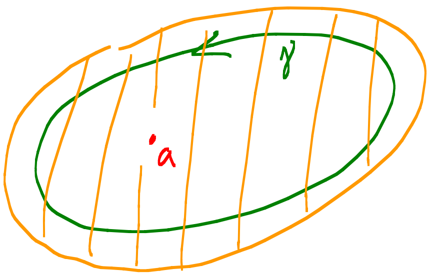
Lesson 16 The averaging property and maximum principle

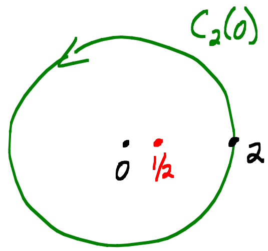
15,16 due Monday, Oct. 1. Exam 1 in class Wed., Oct 3 covering Lessons 1–16



$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

EX:



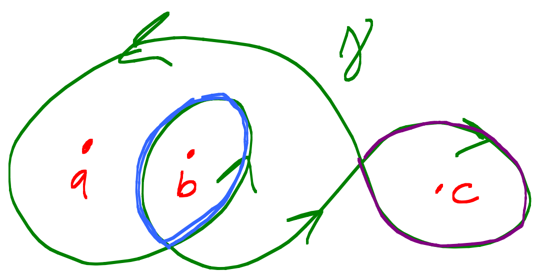
$$\int_{C_2(0)} \frac{e^{2z}}{(z-4)(2z-1)^2}\,dz$$

$$= \int_{C_2(0)} \frac{e^{2z}/z-4}{4\,(z-\frac{1}{2})^2}\,dz = \frac{2\pi i}{1!}\,f'(\tfrac{1}{2})$$

$f(z)$

EX: $\displaystyle\int_{C_2(0)} \frac{e^{2z}}{z^2-1}\,dz$ Aha! $\displaystyle\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$, $A = 1/2$, $B = 1/2$

EX:



Cauchy's Thm

$$\int_\gamma \frac{e^{2z}}{z-a}\,dz = 0 - 0 + 2\pi i\, e^{2a}$$

$f(z)$

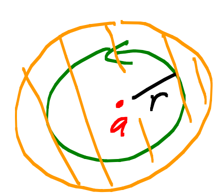
$$\int_\gamma \frac{e^{2z}}{z-b}\,dz = 2\pi i\, e^{2b} + 2\pi i\, e^{2b} - 0$$

$$\int_\gamma \frac{e^{2z}}{z-c}\,dz = 0 + 0 - 2\pi i\, e^{2c}$$

clockwise!

Big Fact: Analytic fcns (and harmonic fcns too!) satisfy the averaging property:

$$(*)\quad f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$



f analytic

$C_r(a)$: $z(t) = a + re^{it}$, $0 \le t \le 2\pi$

$z'(t) = ire^{it}$

or $\underbrace{\frac{1}{2\pi r}}_{\text{circumference}} \int_0^{2\pi} f(a+re^{it}) \underbrace{(r\,dt)}_{ds}$

<u>Consequence</u>: u harmonic. Get harm conj v. $f = u + iv$ analytic

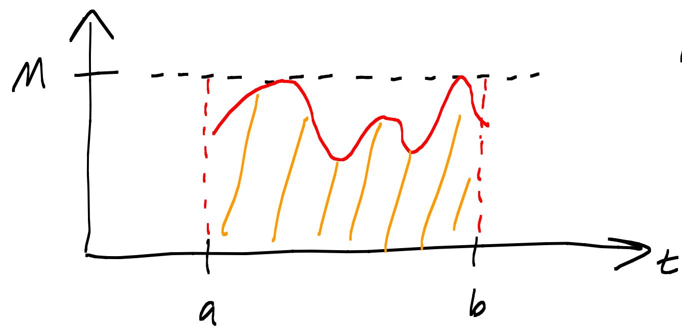
Re $(*)$ gives ave prop for u!

<u>Why</u>: $$f(a) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(z)}{z-a}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{it})}{[a+re^{it}]-a} \underbrace{(ire^{it}\,dt)}_{dz = z'(t)\,dt}$$

$$= \text{ave} \checkmark$$

<u>Fact</u>: Ave prop $\Rightarrow$ Max Princ.

<u>Calculus Lemma</u>:



$h(t)$ continuous

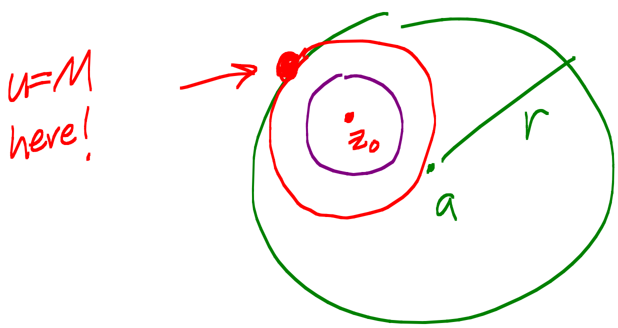
$h(t) \leq M$.

If $\int_a^b h(t)\,dt = M(b-a)$, then $h(t)$ must be $\equiv M$.

<u>Max princ #1 for harmonic fcns</u>: Suppose u is harmonic on $D_r(a)$ and continuous on $\overline{D_r(a)}$. Then

$$\max_{\overline{D_r(a)}} u = \max_{C_r(a)} u$$

The max is attained on the boundary.



<u>Fact:</u> u, a continuous fcn on a closed bounded set $\overline{D_r(a)}$, attains its max, say at z_0. If $z_0 \in C_r(a)$, then ✓.

$M = \text{Max}\, u = u(z_0) =$ ave over purple circle $\longrightarrow$ ave over red circle

Aha! Calc Lemma $\Rightarrow$ $u \equiv M$ on red circle

<u>Important fact:</u> If f is analytic on a disc $D_r(a)$ and $|f|$ is <u>constant</u> there, then f must be <u>constant</u>.

<u>Why:</u> $f(x+iy) = u(x,y) + iv(x,y)$. $\qquad |f|^2 = u^2 + v^2$

If $|f| \equiv \text{const}$, then $\boxed{u^2+v^2 = c.}$ $\quad (+)$

<u>Case $c=0$:</u> Both $u \equiv 0$, $v \equiv 0$. ✓

<u>Case $c>0$:</u> f analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ C-R Eqns.

$$\begin{aligned} \tfrac{\partial}{\partial x}(+)&: \\ \tfrac{\partial}{\partial y}(+)&: \end{aligned} \begin{cases} 2u\,u_x + 2v\,v_x = 0 \\ 2u\,u_y + 2v\,v_y = 0 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0} \\ \quad -v_x \quad\;\; u_x \qquad\qquad \end{array}$$

<u>not</u> the zero vector! $u^2+v^2>0$

Aha! Det matrix must $= 0$. $\text{Det} = \underbrace{(u_x)^2 + (v_x)^2}_{\equiv 0}$

Red Box #1: $f' = u_x + i v_x \equiv 0$. So $f \equiv$ const.

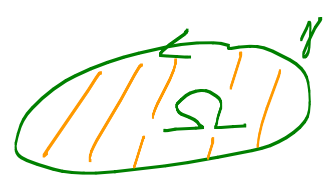
[or C-R eqns show $\nabla u \equiv 0$, $\nabla v \equiv 0 \Rightarrow u, v$ const.]

Theorem: Suppose f is analytic on a domain Ω.

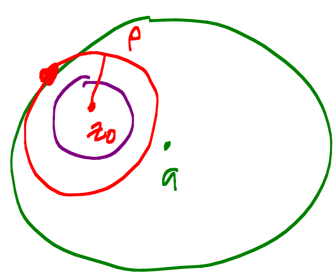
Max Princ #1: If $|f|$ attains a local max inside Ω, then $f \equiv$ const on Ω.

Max Princ #2: If Ω is bounded and f is further assumed to be continuous on $\overline{\Omega}$, then the max modulus of f is attained on the boundary.

$$\max_{\overline{\Omega}} |f| = \max_{\gamma} |f|$$



Why:



$f(z_0) =$ ave $\rightarrow$ ave

$$|f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt \right|$$

$$\underbrace{|f(z_0)|}_{= M} \leq \frac{1}{2\pi} \int_0^{2\pi} \underbrace{|f(z_0 + \rho e^{it})|}_{\leq M}\, dt \leq M$$

So $|f| \equiv M$ on red circle!