Lesson 16 cont'd Dirichlet Problem and Poisson's Formula

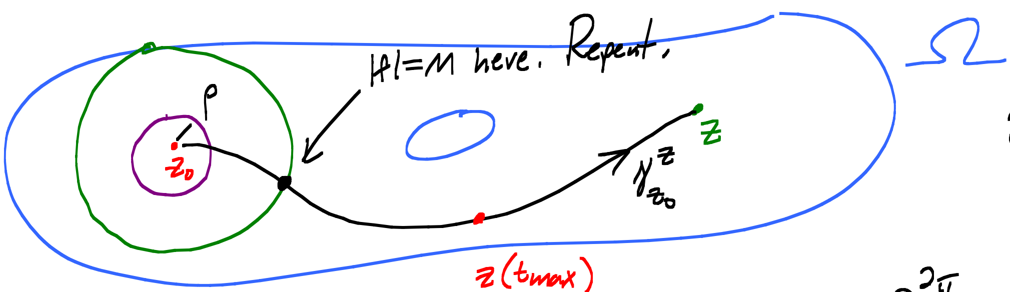
Lessons 15, 16 due Monday. Exam 1 in class Wed., Oct. 3.

Max Princ. for analytic fcns #1: f analytic on a domain Ω.
If $|f|$ has a local max at pt $z_0 \in \Omega$, then $f \equiv$ const on Ω.

True, but not proved in book.

Book's version: Suppose f is analytic on Ω and $|f|$ assumes its max at $z_0 \in \Omega$. Then $f \equiv$ const on Ω.

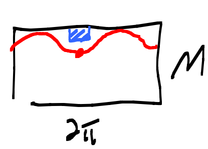
Why: Suppose $|f(z_0)| = M$ at the max.



$\gamma_{z_0}^{z}: z(t)$
$a \le t \le b$

$$f(z_0) = \text{ave on purple circle} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt$$

$$M = |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+\rho e^{it})|}_{\le M}\,dt \le \frac{1}{2\pi}(2\pi M) = M$$

Calculus Lemma  $\Rightarrow |f(z_0+\rho e^{it})| \equiv M$

Aha! Can slide ρ from $\rho = 0$ to biggest ρ possible.

See $|f| \equiv M$ inside green circle. Conclude $f \equiv$ const on disc.

Let $t_{max} = \sup\{t : |f(z(t))| = M \text{ for } a \le t \le t_{max}\}$

Continuity shows that $|f(z(t_{max}))| = M$ too. Can repeat max princ arg at $z(t_{max})$. t_{max} not max ↯.

t_{max} must be b. See $f(a) = f(b)$.

Question: Is there a minimum princ for $|f|$?

Oops! Not if f has zeroes. Ex: $f(z) = z^3$ on $D_1(0)$.

Fact: If f has no zeroes on Ω, there is a minimum modulus theorem. [Apply max princ to $1/f$.]

Harmonic fcns more straightforward: Max princ for $-u$

$\Rightarrow$ min princ for u.

Liouville's Theorem for harmonic fcns: If u is harmonic on $\mathbb{C}$ and $u(z) \leq M$ for all $z \in \Omega$, then $u \equiv$ const on $\mathbb{C}$.

Pf: Let v be a harm conj for u. Then $f = u + iv$ is entire.

Trick: $\underbrace{e^{f}}_{\text{entire}} = e^{u+iv} = e^{u} e^{iv}$

$$|e^f| = e^u \leq e^M \quad \text{on } \mathbb{C}.$$

So Liouv. $\Rightarrow$ $e^f \equiv$ const, c.

$$|e^f| = e^u = |c|$$

So $u = \ln|c|$, a const.

Dirichlet problem: Suppose φ is a continuous real valued fcn on the unit circle. Find a real valued fcn u on $\overline{D_1(0)}$

that is continuous on $\overline{D_1(0)}$, harmonic on $D_1(0)$, and

$u(e^{i\theta}) = \varphi(e^{i\theta})$ on the unit circle.

$u(z)$ = Steady state temp at z with temp on $C_1(0)$ given by φ.

Poisson's Formula for the solution:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,\varphi(e^{i\theta})\,d\theta$$

Why: Play around with the Cauchy Integral formula to be able to take the real part.

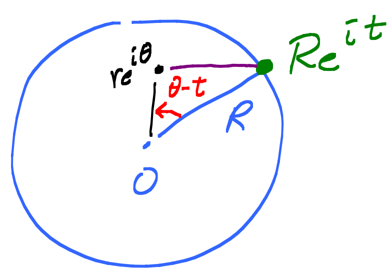
Remark: Max Princ shows that the sol^n to the D. Prob is unique.

Why: Suppose u_1 and u_2 solve D. Prob with bndry data φ.

Then u_1-u_2 is harm on $D_1(0)$, cont on $\overline{D_1(0)}$, and $\equiv 0$ on $C_1(0)$. Max Princ $\Rightarrow$ $u_1-u_2 \le 0$ on $\overline{D_1(0)}$.

Repeat for u_2-u_1. Get $u_2-u_1 \le 0$ too. So $u_2-u_1 \equiv 0$ and $u_1 \equiv u_2$. ✓

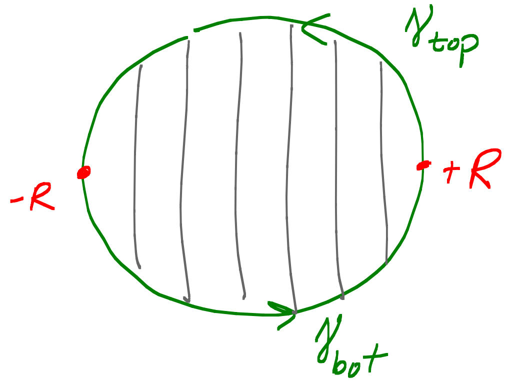
Poisson Formula for $D_R(0)$:



$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(t-\theta)}\,\varphi(Re^{it})\,dt$$

purple line dist squared by law of cosines.

How hard is Green's Thm?



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$$\int_\gamma u\,dx = \int_{\gamma_{bot}} + \underbrace{\int_{\gamma_{top}}}_{-\int_{-\gamma_{top}}}$$

$$\gamma_{bot} = \begin{cases} x = t \\ y = -\sqrt{R^2 - t^2} \end{cases} \quad -R \le t \le R$$

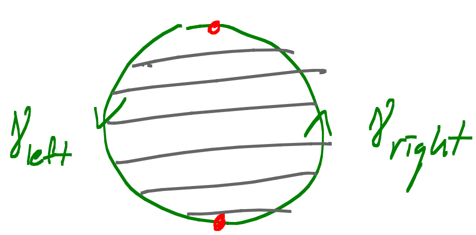
$$\underbrace{\int_{-R}^{R} u(t, -\sqrt{R^2-t^2})\,dt}_{\int_{\gamma_{bot}}} \;-\; \int_{-R}^{R} u(t, \sqrt{R^2-t^2})\,dt$$

$$= \int_{-R}^{R} \underbrace{\Big[u(bot) - u(top)\Big]}_{-\int_{bot}^{top} \frac{\partial u}{\partial y}\,dy} dt = -\int\left(\int \frac{\partial u}{\partial y}\,dy\right) dx$$

$$= -\iint_{D_1(0)} \frac{\partial u}{\partial y}\,dx\,dy$$

Other half:

$$\int_\gamma v\,dy$$



$$= \int_{-R}^{R}\left(\int \frac{\partial v}{\partial x}\,dx\right) dy \quad \checkmark$$