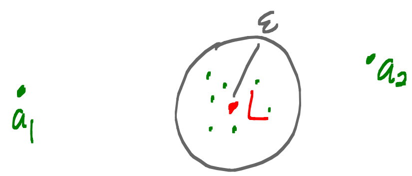
Lesson 19 Power Series

Lessons 19, 20, 21 due Wed. Oct 17

Read 5.1 and 5.2 carefully

Sequence $\{a_n\}_{n=1}^{\infty}$ converges to L means, given any $\varepsilon > 0$, there is an N such that $|a_n - L| < \varepsilon$ if $n > N$. Say $\lim\limits_{n\to\infty} a_n = L$.



Fact: All basic facts about real limits are true for complex limits and the proofs are the same.

EX: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim a_n}{\lim b_n}$ if those limits exist and $b_n \to L \neq 0$

Power series: $f(z) = \sum\limits_{n=0}^{\infty} a_n (z-a)^n$ centered at a.

EX: Geometric series: $1 + z + z^2 + \cdots + z^N = S_N$

$$zS_N = z + z^2 + \cdots + z^N + z^{N+1}$$

$$S_N - zS_N = 1 - z^{N+1}$$

$$S_N = \underbrace{\frac{1}{1-z}}_{L} - \underbrace{\frac{z^{N+1}}{1-z}}_{\varepsilon(z) \to 0 \text{ if } |z|<1}$$

because $|\varepsilon(z)| = \dfrac{|z|^{N+1}}{|1-z|} \le \dfrac{|z|^{N+1}}{1-|z|}$

$\to 0$ when $|z|<1$ as $N \to \infty$.

$||1-z| \ge \underbrace{\big|1-|z|\big|}_{=1-|z| \text{ if } |z|<1}$

Conclude $\sum_{n=0}^{N} z^n \to \frac{1}{1-z}$ if $|z|<1$.

Error estimate: $\left|\sum_{0}^{N} z^n - \frac{1}{1-z}\right| \leq \frac{r^{N+1}}{1-r}$ if $|z|<r<1$.

uniformly in $D_r(0)$.

Say $\sum_{0}^{N} z^n$ converges uniformly on $D_r(0)$ to $\frac{1}{1-z}$ when $0<r<1$,

Note: It converges, but not uniformly, to $\frac{1}{1-z}$ on $D_1(0)$.

Because $|\mathcal{E}_N(z)| = \frac{|z|^{N+1}}{|1-z|}$ ← Boom! at $z=1$.

Fact: If $\sum_{n=1}^{\infty} u_n$ converges, then $u_n \to 0$ as $n\to\infty$.

Note: One way $\Longrightarrow$ implication. $\left[\sum_{n=1}^{\infty} \frac{1}{n} = \infty\right]$

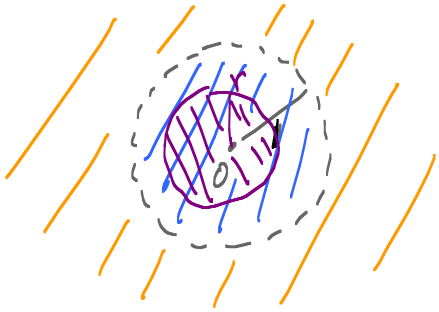
Why: $\sum_{n=1}^{\infty} u_n = L$. $\quad S_N = \sum_{n=1}^{N} u_n$

$$\begin{array}{ccccc} u_N & = & S_N & - & S_{N-1} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & = & L & - & L \end{array} \quad \text{as } N\to\infty$$

✓

Geometric Series: $\sum_{0}^{\infty} z^n$. $\quad u_n = z^n$. $\quad |u_n| = |z|^n$ do not $\to 0$ when $|z|\geq 1$.

So Geometric Series diverges for $|z|\geq 1$.



Convergence $\boxed{////}$

Divergence $\boxed{///}$

Uniform convergence $\boxed{\backslash\backslash\backslash\backslash}$ $0<r<1$

$R=1$ is the "radius of convergence" about $z=0$.

Def^n: A sequence of functions $f_n(z)$ on a domain Ω converges uniformly on compact subsets to $f(z)$ if

for each compact set $K \subset \Omega$
and $\varepsilon > 0$,

there is an N such that $|f_n(z) - f(z)| < \varepsilon$

for all $z \in K$.

uniform on K

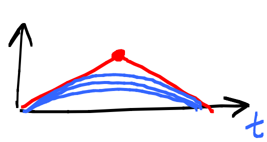
Notation $f_n \twoheadrightarrow f$

Note: Compact sets in $\mathbb{C}$ are the closed and bounded sets.

Remark: When $\Omega = D_1(0)$. Unif conv on compacts $\iff$ unif conv on $D_r(0)$ for each $0<r<1$.

Fantastic complex variable fact: Morera's Theorem $\Longrightarrow$ If a sequence f_n of analytic fcns on a domain Ω converges uniformly on compact subsets to f, then f is analytic!

Way false $\mathbb{R} \to \mathbb{R}$:



blue are smooth
$\longrightarrow$ unif to red.
red not diff'ble!

Weierstraß example: $\underbrace{\sum_{n=1}^{N} \frac{1}{2^n} \sin(2^n t)}_{C^\infty \text{ smooth}} \longrightarrow$ unif to a cont fcn

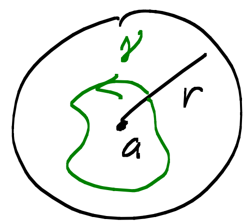
Nowhere diff'le !

Why Fantastic Fact: We can let $\Omega = D_r(a)$.

Fact: If continuous fcns conv uniformly, the limit is continuous.

We assume $\underset{\uparrow \text{analytic}}{f_n} \rightrightarrows f$ on $D_r(a)$. Then f is continuous.

[Morera's: If f is continuous, and $\int_\gamma f\, dz = 0$ for every closed γ in the domain, then f is analytic.



tr(γ) is closed and bounded, hence compact

$$\int_\gamma f\, dz = \lim_{n\to\infty} \int_\gamma \underset{\uparrow \text{analytic}}{f_n}\, dz = 0 \text{ by Cauchy's Thm!}$$

Why can we $\int \lim = \lim \int$. $\leftarrow$ need unif conv for this.

$$\left| \int_\gamma f\, dz - \int_\gamma f_n\, dz \right| = \left| \int_\gamma (f - f_n)\, dz \right|$$

$$\leq \underbrace{\left(\max_{tr(\gamma)} |f - f_n| \right)}_{< \varepsilon \text{ if } n > N.} \cdot \text{Length}(\gamma) \quad \checkmark$$

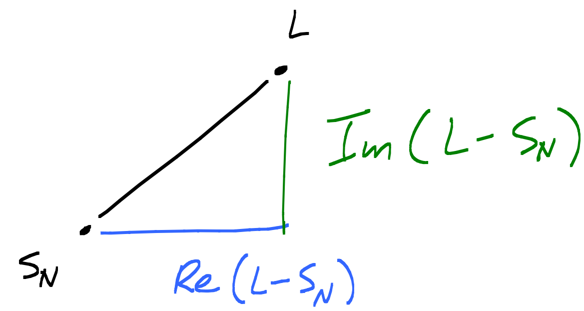
Fact: Absolutely convergent series converge!

$\sum_{n=1}^{\infty} a_n$ converges absolutely : Means that $\sum_{n=1}^{\infty} |a_n| < \infty$.

Show $\sum_{n=1}^{\infty} a_n$ converges.

Assume real variable version.

Assume $S_N \to L$.



$|\text{Legs}| \leq |\text{Hyp}|$

Easy to show real variable fact $\Longrightarrow$ complex version.