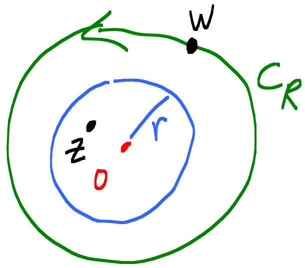
Lesson 20 Cauchy Integral and power series

19, 20, 21 due Wed.

Assume f analytic on a disc bigger than $D_R(0)$



$0 < r < R$. $z \in D_r(0)$

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w - z}\, dw$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{|z|}{R} < \underbrace{\frac{r}{R}}_{<1}$$

$$= \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w} \underbrace{\left(\frac{1}{1 - \frac{z}{w}} \right)}\, dw$$

$$\frac{1}{1-\frac{z}{w}} = \left[1 + \left(\frac{z}{w}\right) + \cdots + \left(\frac{z}{w}\right)^N \right] + \underbrace{\frac{\left(\frac{z}{w}\right)^{N+1}}{1 - \frac{z}{w}}}_{\mathcal{E}_N\left(\frac{z}{w}\right)}$$

$$= \underbrace{\left(\sum_{n=0}^{N} \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w} \cdot \frac{z^n}{w^n}\, dw \right)}_{\sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w^{n+1}}\, dw \right)}_{\frac{1}{n!} f^{(n)}(0)} z^n} + \underbrace{\frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w} \mathcal{E}_N\left(\frac{z}{w}\right) dw}_{E_N(z)}$$

Finally,

$$|E_N(z)| \le \frac{1}{2\pi} \left(\max_{C_R} \frac{|f|}{R} \right) \underbrace{\left(\max_{|w|=R} \left| \mathcal{E}_N\left(\frac{z}{w}\right) \right| \right)}_{\le \frac{\left(\frac{r}{R}\right)^{N+1}}{1 - \frac{r}{R}}} \underbrace{\text{Length}(C_R)}_{2\pi R}$$

$\longrightarrow 0$ uniformly in $z \in D_r(0)$ as $N \to \infty$. !

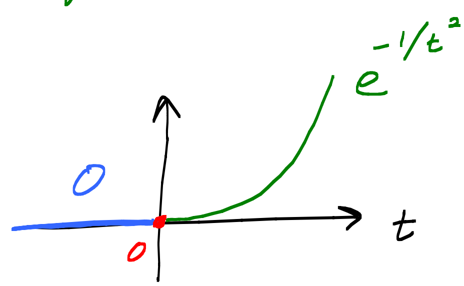
Ex: $f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

converges uniformly on $D_r(0)$ for any $r > 0$.

Say Radius of convergence $= \infty$.

Fact: Analytic functions are given locally by convergent power series!

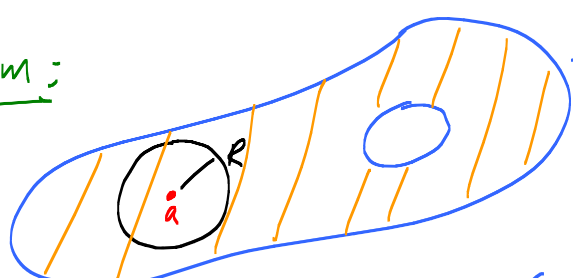
Way false $\mathbb{R} \to \mathbb{R}$:



$$h(t) = \begin{cases} 0 & t \le 0 \\ e^{-1/t^2} & t > 0 \end{cases}$$ is a C^∞-smooth fcn.

Taylor Series at $0 \equiv 0 \neq h(t)$ near 0!

Theorem:



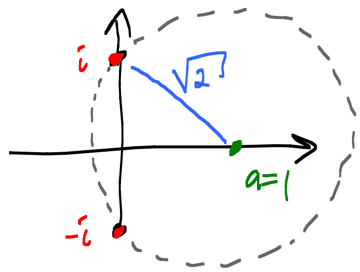
Ω domain, f analytic on Ω

Then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$

converges uniformly on $D_r(a)$ when $0 < r < R < \text{dist}(a, b\Omega)$.

Cor: Radius of convergence of power series for $f(z)$ about a is $\ge \text{dist}(a, b\Omega)$

Ex: $f(z) = \frac{1}{z^2+1}$ at $a=1$. $R = ?$



$\Omega = \mathbb{C} - \{\pm i\}$

CoR $\Rightarrow R \geq \sqrt{2}$

Aha! $R \leq \sqrt{2}$ because f blows up at $\pm i$.

(Power series converge to analytic fcns inside R of C and analytic fcns are continuous. They don't blow up.)

So $R = \sqrt{2}$.