Lesson 22: Radius of convergence

19, 20, 21 due Wed.

Sup = supremum = "least upper bound"

Big real analysis fact: A set in $\mathbb{R}$ that is bounded from above has a least upper bound. ← Equiv to completeness.

Ex: $\sup\{q \in \mathbb{Q} : q^2 < 2\} = \sqrt{2}$

Hadamard's Formula: $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ has R. of C. given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$ ← Cases $R = 0, \infty$ good.

Def^n: $$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \underbrace{\sup\{\sqrt[n]{|a_n|} : n \geq N\}}$$

non-increasing in N
bounded from below by 0.
So limit exists.

equiv to completeness too!

Why do power series have a radius of conv?

Hmmm. If $\sum_n a_n w^n$ converges at w, then

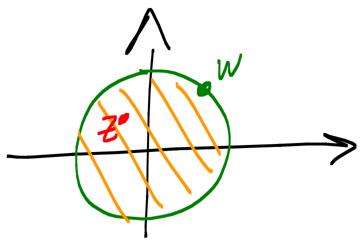
$a_n w^n \to 0$ as $n \to \infty$. Taking $\varepsilon = 1$, we can say $\exists N$ such that $|a_n w^n| < 1$ if $n > N$.

Now, if $|z| < |w|$, then

$$|a_n z^n| = \left|a_n w^n \cdot \frac{z^n}{w^n}\right| = \underbrace{|a_n w^n|}_{<1} \left|\frac{z}{w}\right|^n < \left|\frac{z}{w}\right|^n$$

Aha! $\left|\frac{z}{w}\right| < 1$.

Comparison with conv geom series shows $\sum a_n z^n$ converges!



Notice: We showed that if $|a_n r^n| < M$ for $n > N$,

then $\sum a_n z^n$ converges for $|z| < r$.

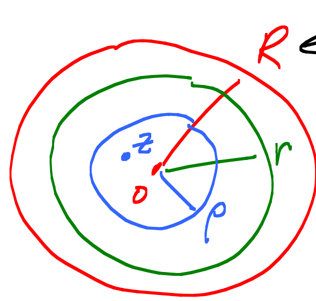
Also, if $|a_n r^n|$ not bounded, then $|a_n z^n| \not\to 0$

when $|z| = r$. So series diverges.

Fact: $R = \sup\{r : |a_n r^n| \text{ is a bounded seq.}\}$

Not a max.

Pf of Hadamard's:



$\leftarrow$ from Hadamard's

Then $\frac{1}{R} < \frac{1}{r} < \frac{1}{\rho}$ $\leftarrow$ $\sup\{\sqrt[n]{|a_n|} : n \geq N\}$ decreasing to $\frac{1}{R}$

So $\exists N$ such that $\sup_{n \geq N} \{\sqrt[n]{|a_n|}\} < \frac{1}{r}$

So $\sqrt[n]{|a_n|} < \frac{1}{r}$ for $n \geq N$.

$$\underline{|a_n| < \frac{1}{r^n}} \text{ for } n \geq N$$

Hmmm. If $|z| < \rho$, then

$$|a_n z^n| = |a_n||z|^n < \frac{1}{r^n}|z|^n < \Big(\underbrace{\frac{\rho}{r}}_{<1.}\Big)^n$$

Aha! $\sum a_n z^n$ converges by comparison with geom series!

So R.of.C is $\geq R$!

Last step: show $\sum a_n z^n$ <u>diverges</u> if $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \underbrace{\sup_{n \geq N} \sqrt[n]{|a_n|}}$$

So, for each N, there is an $n \geq N$ such that

$$\frac{1}{|z|} < \frac{1}{R} \leq \sqrt[n]{|a_n|}$$

mult. by $|z|^n$ $\longrightarrow$ $\frac{1}{|z|^n} < \frac{1}{R^n} \leq |a_n|$

$$\underline{1} < |a_n z^n| \longleftarrow \text{terms can't go to zero!}$$

So series diverges if $|z| > R$! So R. of C. $= R$.

Huge Facts:

1) Power series converge uniformly on $D_r(a)$ when $0 < r < R$.

Consequently, the limit is analytic.

2) Can diff'tiate power series term by term.

Why: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f' has a conv power series

$$f'(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \quad \text{where}$$

$$b_n = \frac{(f')^{(n)}(a)}{n!} = \frac{f^{(n+1)}(a)}{n!} = \frac{(n+1)\, f^{(n+1)}(a)}{\underbrace{(n+1)\, n!}_{(n+1)!}}$$

$$= (n+1) \frac{f^{(n+1)}(a)}{(n+1)!} = (n+1)\, a_{n+1} \quad \text{where}$$

$$f(z) = \sum_0^{\infty} a_n (z-a)^n$$

$$b_n z^n = \frac{d}{dz}\left[a_{n+1} z^{n+1}\right] \checkmark$$

3) R. of C for series for f' = R. of. C for f.

Can solve ODE's!

EX: $$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$

Step 1: Let x be complex!

$$f''(z) - z f'(z) - f(z) = 0 \leftarrow \text{want}$$

Step 2: Assume f = power series. $= \sum_0^\infty a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$

$$f'(z) = 0 + a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

$$f''(z) = 0 + 0 + 2a_2 + 3\cdot 2 a_3 z + 4\cdot 3 a_4 z^2 + \cdots$$

$$- \; z f'(z) = \qquad a_1 z + 2a_2 z^2 + \cdots$$

$$- \; f(z) = \qquad a_0 + a_1 z + a_2 z^2 + \cdots$$

$$\underbrace{(2a_2 - a_0)}_{a_2 = \frac{1}{2}a_0} + (3\cdot 2 a_3 - a_1 - a_1) z + (4\cdot 3 a_4 - 2a_2 - a_2) z^2$$

$$+ \cdots + \Big[\underbrace{(n+2)(n+1)a_{n+2} - n a_n - a_n}_{\text{recursion formula}}\Big] z^n + \cdots$$

$$= 0$$

↑ want

Taylor's Formula says coeff need to be zero.

How to get a's: $f(0) = a_0$, $f'(0) = a_1$ ← initial conditions!

$$(n+2)(n+1)\,a_{n+2} - n\,a_n - a_n = 0$$

$$\boxed{a_{n+2} = \frac{a_n}{(n+2)}}$$