Lesson 23 Zeroes of analytic functions

$$f''(z) - zf'(z) - f(z) = 0$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad -f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad -zf'(z) = \sum_{n=1}^{\infty} n a_n z^n$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \qquad f''(z) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

want

$$0 = \underbrace{(2\cdot 1\, a_2 - a_0)}_{\text{straggler term } = 0} + \sum_{n=1}^{\infty} \Big[\underbrace{(n+2)(n+1)a_{n+2} - na_n - a_n}_{=0} \Big] z^n$$

$$\boxed{a_{n+2} = \frac{a_n}{(n+2)}} \quad n \geq 1$$

Recursion formula

$$\text{IC} \begin{cases} a_0 = f(0) \\ a_1 = f'(0) \end{cases}$$

$$a_2 = \tfrac{1}{2} a_0$$

$n=1$: $a_3 = \dfrac{a_1}{3}$

$n=2$: $a_4 = \dfrac{a_2}{4} = \dfrac{a_0}{4\cdot 2}$

$n=3$: $a_5 = \dfrac{a_3}{5} = \dfrac{a_1}{5\cdot 3}$

$\vdots$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = a_0 + a_1 z + \tfrac{1}{2} a_0 z^2 + \tfrac{1}{3} a_1 z^3 + \cdots$$

$$= a_0\left(1 + \tfrac{1}{2}z^2 + \tfrac{1}{4\cdot 2} z^4 + \tfrac{1}{6\cdot 4\cdot 2} z^6 + \cdots\right) + a_1\left(z + \tfrac{1}{3}z^3 + \tfrac{1}{5\cdot 3} z^5 + \tfrac{1}{7\cdot 5\cdot 3} z^7 + \cdots\right)$$

What is the R. of C.? Hmmm.

Ratio Test: $\lim\limits_{\text{way out}\to\text{end}} \left| \dfrac{\text{Term way out there}}{\text{Term before}} \right|$

$$= \lim_{n\to\infty} \frac{|a_{n+2} z^{n+2}|}{|a_n z^n|} = \lim_{n\to\infty} \underbrace{\left|\frac{a_{n+2}}{a_n}\right|}_{\frac{1}{n+2} \text{ from recursion!}} |z|^2 = 0 < 1$$

Two solutions: $R = \infty$.

Philosophy: Analytic fcns are like polys.

Zeroes of analytic fcns: $f(z) = \sum_{n=0}^{\infty} a_n z^n$. R of C > 0

Suppose $f(0) = 0$. Then $a_0 = \frac{f(0)}{0!} = 0$.

Case 1: All a_n's $= 0$. Then $f(z) \equiv 0$.

Case 2: Not all a_n's $= 0$. Then there is a first non-zero a_n. Say $a_N \neq 0$. $a_n = 0$ for $n = 0, 1, 2, \ldots, N-1$.

Then $f(z) = a_N z^N + a_{N+1} z^{N+1} + \cdots$

$$= z^N \left[\underbrace{a_N + a_{N+1} z + a_{N+2} z^2 + \cdots}_{F(z)} \right]$$

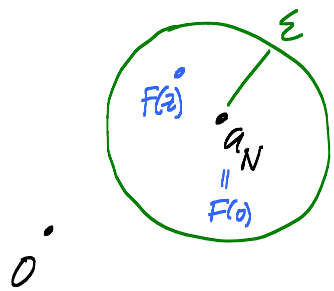
Aha! Power series for f converges for exactly same z as power series for $F(z)$. They have same R. of C.!

$$\underbrace{f(z)}_{\text{analytic}} = z^N \underbrace{F(z)}_{\text{analytic}} \qquad \text{Note: } F(0) = a_N \neq 0.$$

F analytic $\Rightarrow$ F continuous. $F(0) \neq 0 \Rightarrow$

F non-vanishing on a disc $D_\delta(0)$ for some $\delta > 0$.

Why:



$\varepsilon = \dfrac{|a_N|}{2}$ $\qquad F(0) = a_N$

$|F(z) - F(0)| < \varepsilon$

if $|z - 0| < \delta$.

Big consequence: If $f \not\equiv 0$, then the zeroes of f must be isolated [$f(0) = 0$. There is a disc $D_\delta(0)$ such $z=0$ is the only zero of f in $D_\delta(0)$.] just like polys.

And $\qquad f(z) = z^N F(z), \quad F(0) \neq 0$

$z = 0$ is a zero of multiplicity N.

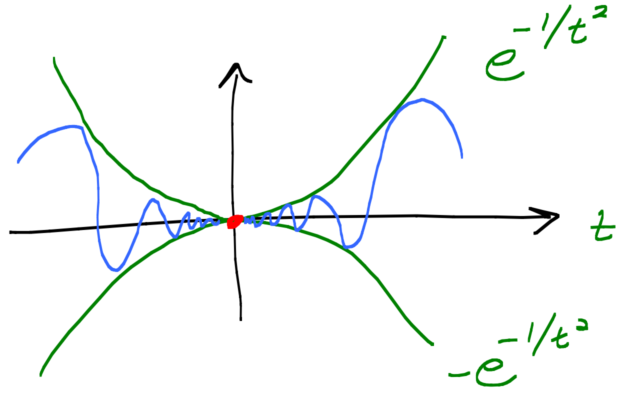
$a_n = \dfrac{f^{(n)}(0)}{n!}$. $\qquad f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots, N-1$

$f^{(N)}(0) \neq 0 \leftarrow$ mult $= N$.

Zeroes of analytic fcns cannot pile up (accumulate).

Way false $\mathbb{R} \to \mathbb{R}$:

$$\begin{cases} e^{-1/t^2} \cdot \sin\frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases} \quad C^\infty \text{ smooth}$$



zeroes pile up at $t=0$!

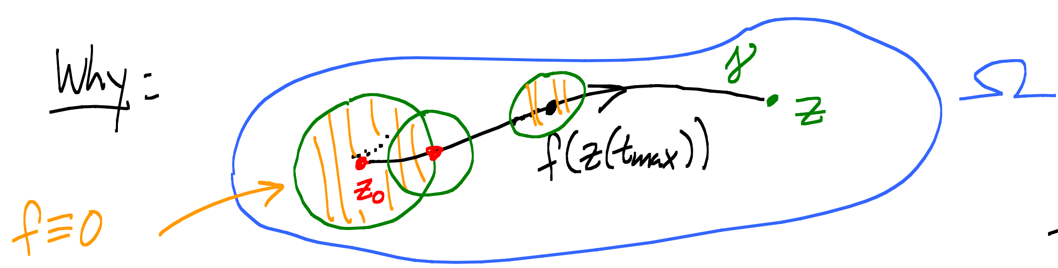
Def^n: $z_0 \in \Omega$ is called an accumulation point of a set $S' \subset \Omega$ if there is a sequence $z_n \in S'$ such that $z_n \to z_0$ as $n \to \infty$ and $z_n \neq z_0$ for each n.

Same as saying: For each little disc $D_\varepsilon(z_0) \subset \Omega$,

$$\left(D_\varepsilon(z_0) - \{z_0\}\right) \cap S' \neq \emptyset.$$

Big fact: Given an analytic fcn f on a domain Ω, if $\{z : f(z) = 0\}$ has an accumulation point $z_0 \in \Omega$, then $f \equiv 0$ on Ω. [Identity Thm.]

Why:



Let $t_{max} =$

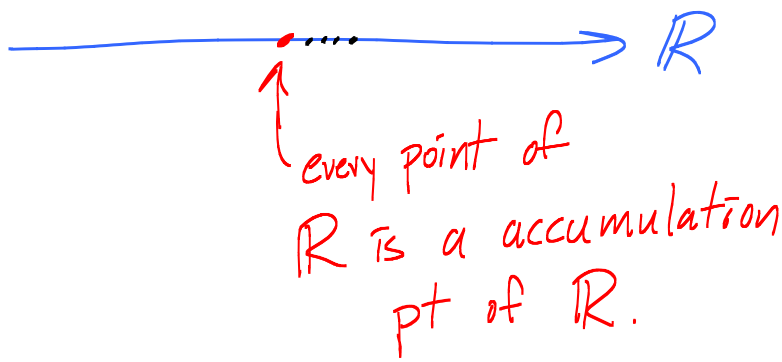
$f(z(t)) = 0$

for $a \leq t \leq t_{max}$

Continuity $\Rightarrow f(z(t_{max})) = 0$ too! Get disc where $f \equiv 0$. t_{max} not max. ↯. $t_{max} = b$.

<u>Consequence</u>: e^z is the <u>only</u> extension of e^x to $\mathbb{C}$ as an analytic fcn.

<u>Why</u>: Suppose $E(z)$ does it too. $e^z - E(z) \equiv 0$ on $\mathbb{R}$



Identity Thm $\Rightarrow e^z - E(z) \equiv 0$ on $\mathbb{C}$.