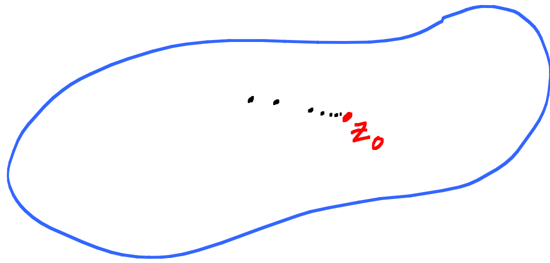
Lesson 24 Laurent series

22, 23, 24 due Wed.

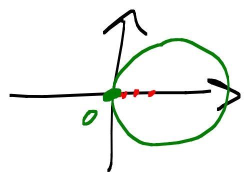
Identity Theorem:
or
(Uniqueness Thm)

Ω 

Z = zero set of analytic f on domain Ω.

If Z has an accumulation pt (= limit pt) $z_0 \in \Omega$, then $f \equiv 0$ on Ω.

EX: $f(z) = \sin\frac{1}{z}$ on $D_1(1)$.



OK for zeroes to pile up at a boundary point.

Cor: If an analytic fcn f vanishes on a disc $D_\varepsilon(z_0) \subset \Omega$, then $f \equiv 0$ on the domain of def^n Ω.

Cor: If f and g are analytic on a domain Ω and $f = g$ on set with a limit pt in Ω, then $f \equiv g$.

Application: All trig identities hold for complex angles.

Why: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\underbrace{\sin 2z} = \underbrace{2\sin z\cos z} \qquad \Omega = \mathbb{C}$$

Two analytic fcns that agree on $\mathbb{R} \subset \mathbb{C}$.

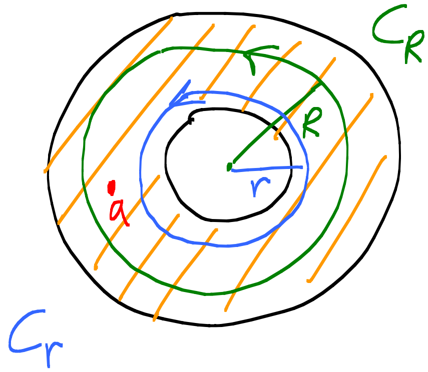
every pt in $\mathbb{R}$ is a lim. pt of $\mathbb{R}$

EX: $\cos(\underset{w}{\alpha} + \underset{z}{\beta}) = \cos\underset{w}{\alpha}\cos\underset{z}{\beta} - \sin\underset{w}{\alpha}\sin\underset{z}{\beta}$

First let β go complex. Use Id. Thm. Then let α go complex. Use Id Thm a second time.

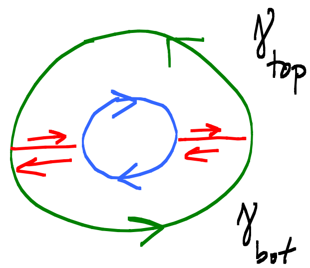
Cauchy Theorem on an annulus:

f analytic on bigger annulus.



Then $$\left(\int_{C_R} - \int_{C_r}\right) f\, dz = 0.$$

Why:



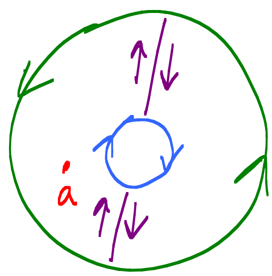
$$0 = \left(\underbrace{\int_{\gamma_{top}}}_{=0} + \underbrace{\int_{\gamma_{bot}}}_{=0}\right) f\, dz = \left(\int_{C_R} - \int_{C_r}\right)$$

$=0$ by Cauchy's

Cauchy Theorem:

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\, dz$$

Why:



$$\underbrace{\frac{1}{2\pi i}\int_{\gamma_{left}}}_{= f(a) \text{ by C.I.F.}} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_{right}}}_{=0 \text{ by Cauchy's}}$$

Laurent expansions:

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w} \frac{1}{1-\frac{z}{w}}\, dw + \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{z}\left[\frac{1}{1-\frac{w}{z}}\right] dw$$

$|\frac{w}{z}| < 1$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w^{n+1}}\, dw \right)}_{a_n} z^n + \underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{z} \left[1 + \left(\frac{w}{z}\right) + \left(\frac{w}{z}\right)^2 + \cdots \right] dw}_{\sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int f(w) w^n\, dw \right)}_{a_{-(n+1)}} \frac{1}{z^{n+1}}}$$

Power series!

$$= \sum_{n=-\infty}^{\infty} a_n z^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{w^{n+1}}\, dw .$$

$r < \rho < R$

Note: Cauchy Thm $\Rightarrow$ $\displaystyle\int_{C_R} = \int_{C_\rho} = \int_{C_r} \frac{f(w)}{w^{n+1}}\, dw$

Fact: Laurent series converge uniformly on compact sub-annuli.

Exciting fact: If f is analytic on $D_r(z_0) - \{z_0\}$

Isolated singularity at z_0

then Laurent series is valid on $D_r(z_0) - \{z_0\}$!

Huge consequence: Only 3 possible behaviors at an isolated sing.

1) $\displaystyle f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ $\longleftarrow$ no negative terms.

f = conv. power series. If we set $f(z_0) = a_0$, then

f analytic on $D_r(z_0)$! z_0 is a removable sing.

2) $$f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-(N-1)}}{(z-z_0)^{N-1}} + \cdots + \frac{a_{-1}}{z-z_0}}_{\text{Principal part}} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\text{power series}}$$

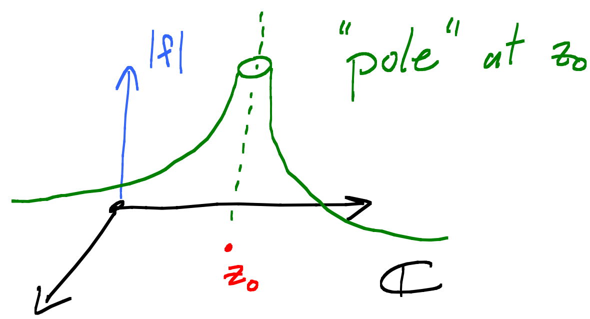
finitely many negative power terms.

f has a pole of order N $(a_{-N} \neq 0)$.

Hmmm. $$f(z) = \underbrace{\frac{1}{(z-z_0)^N}\left[a_{-N} + a_{-N+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{N-1}\right]}_{\sim \frac{a_{-N}}{(z-z_0)^N} \text{ near } z_0.} + G(z)$$

$G(z)$: anal. near z_0

Aha! $\lim_{z \to z_0} f(z) = \infty$



3) $$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$$ ← ∞ many non-zero a_{-n}'s

then f has an essential sing at z_0.

EX: 1) $$\frac{\sin z}{z} = \frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)}{z} = \frac{z\left[1 - \frac{z^2}{3!} + \cdots\right]}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$ ← $R = \infty$

is <u>entire</u> if I set $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$

2) $$\frac{\cos z}{z^3} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - }{z^3} = \underbrace{\frac{1}{z^3} - \frac{1/2!}{z}}_{\text{P.P.}} + \underbrace{\frac{z}{4!} - \frac{z^3}{6!} + \cdots}_{\substack{\text{analytic} \\ \text{near } 0}}$$

$z=0$ is a <u>pole</u> of order 3.

3) $$e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

has an <u>ess. sing</u> at $z=0$.

<u>The most important a_n:</u> $$a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w-w_0)^{n+1}}\, dw$$

Residue of f at w_0 $\longrightarrow$ $$a_{-1} = \frac{1}{2\pi i} \int_{C_\rho} f(w)\, dw$$

$\text{Res}_{w_0} f$

$$\boxed{\int_{C_\rho(w_0)} f\, dw = 2\pi i \, \text{Res}_{w_0} f}$$

Baby Residue Thm.