Lesson 26 Residue integrals

25, 26, 27 due Wed

Def: Suppose f analytic on $\{z : |z| > R\}$. Then we think of f as having an "isolated singularity at ∞." The type of the singularity at ∞ is the type of the sing. of $f(\frac{1}{z})$ at $z=0$.

EX: e^z. Then $f(\frac{1}{z}) = e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \cdots$ ← ess sing at $z=0$

↑ ess sing at ∞.

Fun fact: $f(z)$ is entire:

∞ is removable $\iff$ $f \equiv$ const

∞ is pole $\iff$ f = poly deg $N \geq 1$.

∞ is ess. : All others.

Picard's (little) Thm: If f is entire and not a poly, then hits every value in $\mathbb{C}$ infinitely many times with one possible exception. [e^z misses 0]

Multiplying power series: Just like polynomials!

$$\underbrace{\left(\sum_{n=0}^{\infty} a_n z^n\right)}_{R_a > 0} \underbrace{\left(\sum_{n=0}^{\infty} b_n z^n\right)}_{R_b > 0} = \underbrace{\sum_{n=0}^{\infty} c_n z^n}_{R_c \geq \min(R_a, R_b)}$$

where $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Why: Leibnitz's formula:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$\frac{(fg)^{n}}{n!} = \sum_{k=0}^{n} \frac{n!/n!}{k!\,(n-k)!} f^{(k)} g^{(n-k)}$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Applications of Residue Thm:

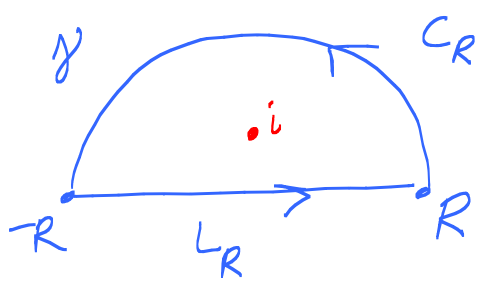
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \leftarrow \text{related to Fourier Transf}$$

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt$$

Step 1: $\cos x = \operatorname{Re} e^{ix}$. Let $f(z) = \dfrac{e^{iz}}{z^2+1}$

Notice that $\dfrac{\cos x}{x^2+1} = \operatorname{Re} f(x)$, $x \in \mathbb{R}$.

Step 2:

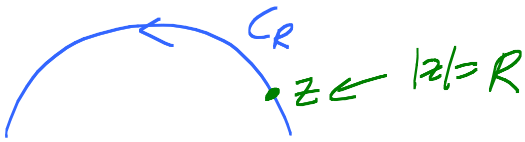


$z^2+1 = 0$ at $z = \pm i$

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_i f(z)$$

Step 3: Show that $\left| \int_{C_R} f\,dz \right| \to 0$ as $R \to \infty$.

$$\left|e^{i(x+iy)}\right| = \left|e^{ix}e^{-y}\right| = e^{-y} \le \underbrace{e^{0}}_{1} \quad \text{when } y \ge 0$$



$$|z^2+1| = |z^2-(-1)| \ge \big||z|^2 - |-1|\big|$$

$$\ge |R^2-1| = R^2-1 \text{ if } R>1$$

So $|f(z)| = \left|\dfrac{e^{iz}}{z^2+1}\right| \le \dfrac{1}{R^2-1}$ if $z \in C_R$ and $R>1$.

<u>And</u> $\left|\displaystyle\int_{C_R} f\,dz\right| \le \left(\max_{C_R} |f|\right) \cdot \text{Length}(C_R) \le \dfrac{1}{R^2-1}\cdot \pi R$

$$\longrightarrow 0 \text{ as } R \to \infty!$$

<u>Step 4</u>: $L_R : z(t) = t, \; -R \le t \le R.$

$z'(t) = 1$

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{e^{it}}{t^2+1}\cdot 1\,dt = \underbrace{\int_{-R}^{R} \frac{\cos t}{t^2+1}\,dt}_{\text{wait!}} + \underbrace{i\int_{-R}^{R} \frac{\sin t}{\underbrace{t^2+1}_{\text{odd!}}}\,dt}_{=0}$$

<u>Step 5</u>:

$$\frac{e^{iz}}{z^2+1} = \frac{1}{z-i}\left[\underbrace{\frac{e^{iz}}{z+i}}_{A_0 + A_1(z-i) + A_2(z-i)^2 + \cdots = F(z)}\right]$$

$$= \frac{A_0}{z-i} + A_1 + A_2(z-i) + \cdots$$

$$\operatorname{Res}_i \frac{e^{iz}}{z^2+1} = A_0 = \frac{F(i)}{0!} = \frac{e^{i(i)}}{i+i} = \frac{e^{-1}}{2i}$$

Step 6:
$$\int_\gamma f\,dz = 2\pi i \operatorname{Res}_i f$$

$$\int_{L_R} f\,dz + \int_{C_R} f\,dz = 2\pi i \frac{e^{-1}}{2i}$$ Let $R\to\infty$

$$\underbrace{\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt}_{I} + i\cdot 0 + 0 = \frac{\pi}{e}$$ ✓

Fact: Suppose f and g are analytic near z_0 and suppose g has a simple zero at z_0.

Then $$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

EX: $$\operatorname{Res}_i \frac{e^{iz}}{z^2+1} \begin{array}{l} \leftarrow f \\ \leftarrow g \end{array}$$

$$\underbrace{\begin{array}{ll} g(z)=z^2+1 & g'(z)=2z \\ g(i)=0 & g'(i)=2i\neq 0 \end{array}}_{i \text{ is a simple zero (order one)}}$$

$$= \frac{f(i)}{g'(i)} = \frac{e^{i(i)}}{2i}$$

Use $$g(z) = \overset{=0}{a_0} + \overset{\frac{g'(z_0)}{1!}}{a_1}(z-z_0) + \cdots = (z-z_0)\underbrace{\left[a_1 + a_2(z-z_0)+\cdots\right]}_{G(z)}$$

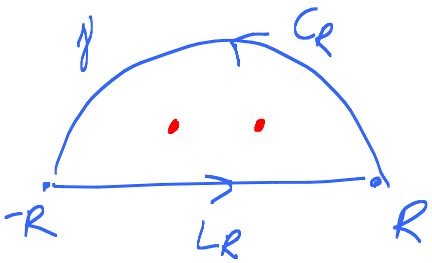
$$\frac{f(z)}{g(z)} = \frac{1}{z-z_0}\left[\frac{f(z)}{G(z)}\right]$$

$A_0 + A_1(z-z_0) + \cdots$

$$\operatorname{Res}_{z_0}\frac{f}{g} = A_0 = \frac{1}{0!}\frac{f(z_0)}{G(z_0)} = \frac{f(z_0)}{a_1} = \frac{f(z_0)}{g'(z_0)} \checkmark$$

Ex: Find $I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

$f(z) = \frac{1}{z^4+1}$ $\leftarrow 1$, $\leftarrow g$



poles at $e^{i\pi/4}, e^{i3\pi/4}$ inside γ

$g(z) = z^4+1$

$g'(z) = 4z^3$

$\begin{cases} g(e^{i\pi/4}) = 0 \\ g'(e^{i\pi/4}) = 4e^{i3\pi/4} \neq 0 \end{cases}$ simple zero

$$\operatorname{Res}_{e^{i\pi/4}}\frac{1}{z^4+1} = \frac{1}{g'(e^{i\pi/4})} = \frac{1}{4e^{i3\pi/4}}$$

$$\operatorname{Res}_{e^{i3\pi/4}}\frac{1}{z^4+1} = \frac{1}{4(e^{i3\pi/4})^3} = \frac{1}{4e^{i9\pi/4}}$$

$$\left|\int_{C_R} f\,dz\right| \le \frac{1}{R^4-1}\cdot\pi R \longrightarrow 0 \quad \text{as } R\to\infty.$$

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R}\frac{1}{t^4+1}\,dt \leftarrow \text{want}$$

$$\underbrace{\int_\gamma f\,dz}_{\int_{L_R} + \int_{C_R}} = 2\pi i \sum_{\substack{\text{inside } \gamma \\ \text{poles}}} \operatorname{Res} f$$

$$\downarrow$$

$$I + 0 = 2\pi i \left[\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right]$$

$$= \frac{\pi i}{2}\left[\underbrace{e^{-i3\pi/4}}_{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i} + \underbrace{e^{-i9\pi/4}}_{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i} \right] = \frac{\pi}{\sqrt{2}}$$