Lesson 27 More Residue integrals

25, 26, 27 due Wed

Exam 2 on Wed. Nov. 14. Final Exam Wed Dec 12 3:30-5:30pm in BRNG 2290

Riemann Removable Singularity Theorem: Suppose is f is analytic and bounded on $D_\varepsilon(z_0) - \{z_0\}$. Then z_0 is removable.

Way false $\mathbb{R} \to \mathbb{R}$: $h(t) = \sin(1/t)$, $t = 0$.

Parallel facts about zeroes and poles:

Zero of order N at z_0:

$f(z_0) = 0, f'(z_0) = 0, \ldots, f^{(N-1)}(z_0) = 0$

but

$f^{(N)}(z_0) \neq 0$.

$$f(z) = a_N (z-z_0)^N + \cdots =$$

$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots \right]}_{F(z),\ F(z_0) = a_N \neq 0}$$

$$\underbrace{f(z) = (z-z_0)^N F(z), \quad F(z_0) \neq 0}_{\text{equiv def}^n \text{ zero, ord } N}$$

Pole of order N:

$$f(z) = \frac{\overset{\neq 0}{A_N}}{(z-z_0)^N} + \cdots + a_0 + a_1(z-z_0) + \cdots$$

$$= \frac{1}{(z-z_0)^N} \underbrace{\left[A_N + A_{N-1}(z-z_0) + \cdots + a_0 (z-z_0)^N + \cdots \right]}_{F(z)}$$

$F(z)$ analytic near z_0

$F(z_0) = A_N \neq 0$

$$\underbrace{f(z) = (z-z_0)^{-N} F(z), \quad F(z_0) \neq 0}_{\text{equiv def}^n \text{ pole ord } N}$$

Last time:

$$\boxed{\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}}$$

when f, g analytic and g has a simple zero at z_0.

Special case: $g(z) = z - z_0$

$$\boxed{\operatorname{Res}_{z_0} \frac{f(z)}{z-z_0} = f(z_0)}$$

EX: $$\operatorname{Res}_i \frac{e^{iz}}{z^2+1} = \operatorname{Res}_i \frac{\left(e^{iz}/z+i\right)}{z-i} = \frac{e^{i(i)}}{i+i} = \frac{e^{-1}}{2i}$$

f

Book: $$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} = \frac{1}{(e^{i\pi/4}-e^{i3\pi/4})(e^{i\pi/4}-e^{i5\pi/4})(e^{i\pi/4}-e^{i7\pi/4})}$$

Ugh!

$$= \frac{f(z_0)}{g'(z_0)} = \frac{1}{4z_0^3}$$ easier!

Fact: Suppose $P(z)$ and $Q(z)$ are polynomials and

1) Q has no zeroes on $\mathbb{R}$

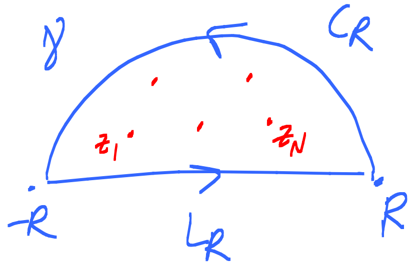
2) $\deg P \leq \deg Q - 2$.

Then $$\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)}\,dt = 2\pi i \sum \operatorname{Res}_{z_k} \frac{P}{Q}$$

z_k zeroes of Q in U.H.P.

$$\left(\text{Also}, \quad \int_{-\infty}^{\infty} \frac{P(t)}{Q(t)} e^{it}\, dt = 2\pi i \sum_{\substack{\text{zeroes } Q \\ \text{in UHP}}} \operatorname{Res}_{z_k} \frac{P(z)}{Q(z)} e^{iz} \right)$$

<u>Why</u>:



<u>Key</u>: Basic polynomial estimate

$$P(z) = a_N z^N + \cdots + a_1 z + a$$

$$A|z|^N \le |P(z)| \le B|z|^N, \quad |z| > R.$$

$$\left| \int_{C_R} \frac{P}{Q}\, dz \right| \le \left(\max_{|z|=R} \left| \frac{P(z)}{Q(z)} \right| \right) (\pi R) \le \frac{B_P\, R^{N_P}}{A_Q\, R^{N_Q}} \pi R$$

$$\le C \frac{1}{R^{\underbrace{N_Q - N_P - 1}_{\ge 1}}} \longrightarrow 0 \text{ as } R \to \infty.$$

<u>Trig Integrals</u>: $I = \int_0^{\underline{\underline{2\pi}}} \cos^6\theta\, d\theta$

Hmmm: $C: z(\theta) = e^{i\theta}, \quad 0 \le \theta \le 2\pi.$ $\quad z'(\theta) = ie^{i\theta} = iz(\theta)$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2} = \frac{1}{2}\left[z(\theta) + \frac{1}{z(\theta)} \right]$$

$$I = \int_0^{2\pi} \left[\frac{z(\theta) + \frac{1}{z(\theta)}}{2} \right]^6 \quad \frac{ie^{i\theta}\, d\theta}{ie^{i\theta}} \quad \begin{array}{l} ie^{i\theta}\,d\theta = dz \\ ie^{i\theta} = iz(\theta) \end{array}$$

$$\boxed{= \frac{1}{2^6} \int_C \left(z + \frac{1}{z} \right)^6 \frac{dz}{iz}}$$

$$= 2\pi i \, \frac{1}{2^6} \operatorname{Res}_0 f(z) \quad \text{where}$$

Res Term!

$$f(z) = \frac{1}{iz}\left(z + \frac{1}{z}\right)^6 = \frac{1}{iz}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + z^{-6}\right)$$

$$\text{So} \quad I = 2\pi i \, \frac{1}{2^6} \cdot \frac{20}{i} = \frac{5\pi}{8}$$

General method: $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

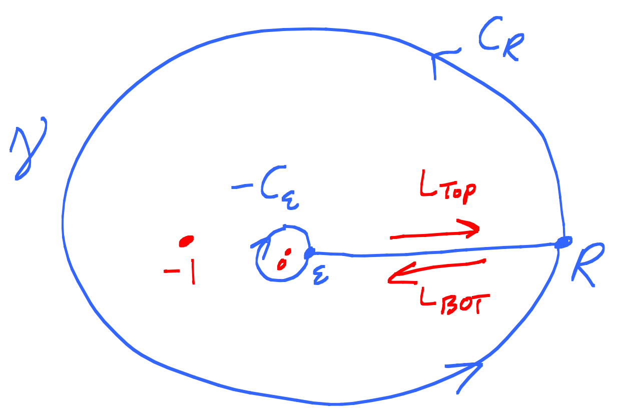
$$= \int_0^{2\pi} R\left(\frac{z(\theta) + \frac{1}{z(\theta)}}{2}, \frac{z(\theta) - \frac{1}{z(\theta)}}{2i}\right) \frac{\overbrace{ie^{i\theta}\, d\theta}^{dz}}{\underbrace{ie^{i\theta}}_{iz(\theta)}}$$

$$= \int_C \underbrace{R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}\right) \frac{1}{iz}}_{f} \, dz$$

$$= 2\pi i \sum \left(\text{Residues of } f \text{ inside } C\right)$$

[Note: Need $R(\cos\theta, \sin\theta)$ not to blow up $0 \le \theta \le 2\pi$.]

EX: $\displaystyle I = \int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\, dx$



$$f(z) = \frac{1}{z^{1/2}(z+1)}$$

where

$$z^{1/2} = e^{\frac{1}{2}\log z} \quad \text{where} \quad \log z = \operatorname{Ln}|z| + i\theta, \quad 0 < \theta < 2\pi$$

$$\theta \in \arg z$$

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{-1} \frac{1}{z^{1/2}(z-(-1))} = 2\pi i \frac{1}{(-1)^{1/2}} = 2\pi i \frac{1}{e^{\frac{1}{2}(\operatorname{Ln}|-1| + i\pi)}}$$

$$= \frac{2\pi i}{\underbrace{e^{i\pi/2}}_{i}} = 2\pi$$

$$\int_{L_{Top}} f(z)\,dz = \int_\varepsilon^R \frac{1}{t^{1/2}(t+1)}\,dt \quad \longleftarrow \text{want}$$

$$\int_{L_{Bot}} f(z)\,dz = -\int_\varepsilon^R \frac{1}{\underbrace{e^{\frac{1}{2}(\operatorname{Ln} t + i2\pi)}}_{\sqrt{t}\,\underbrace{e^{i\pi}}_{-1}}(t+1)}\,dt = \int_\varepsilon^R \frac{1}{t^{1/2}(t+1)}\,dt$$

want!

Last steps: Show $\int_{-C_\varepsilon} f\,dz \to 0$ as $\varepsilon \to 0$ and

$\int_{C_R} f\,dz \to 0$ as $R \to \infty$.

<u>Basic estimate</u>: $\left| z^{1/2} \right| = \left| e^{\frac{1}{2}(\operatorname{Ln}|z| + i\theta)} \right| = |z|^{1/2}$