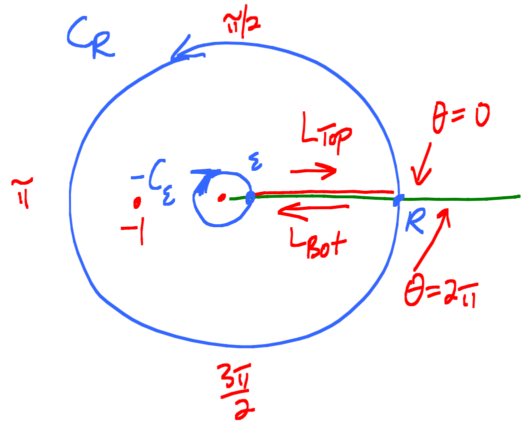
Lesson 28

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$



$$\int_0^{\infty} \frac{1}{\sqrt{x}\,(x+1)}\, dx$$

$$f(z) = \frac{1}{z^{1/2}(z+1)}$$

$$z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\ln|z| + i\theta)}$$

where $\theta \in \arg z$, $0 < \theta < 2\pi$

$$\left| \int_{C_R} f\, dz \right| \le \left(\max_{|z|=R} \frac{1}{|z^{1/2}(z+1)|} \right) \cdot (\pi R) \qquad |z^{1/2}| = |z|^{1/2}$$

$$\le \frac{1}{R^{1/2}\underbrace{(R-1)}_{\text{when } R>1}} \cdot \pi R \qquad \frac{\pi R^{1/2}}{R-1} \to 0 \text{ as } R \to \infty.$$

$$|z+1| = |z-(-1)| \ge \big||z|-1\big| \begin{array}{l} = R-1 \quad \text{when } |z| = R > 1 \\ = 1-\varepsilon \quad \text{when } |z| = \varepsilon < 1 \end{array}$$

$$\left| \int_{-C_\varepsilon} f\, dz \right| \le \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \cdot (\pi\varepsilon) \qquad \frac{\pi\,\varepsilon^{1/2}}{1-\varepsilon} \to 0 \text{ as } \varepsilon \to 0$$

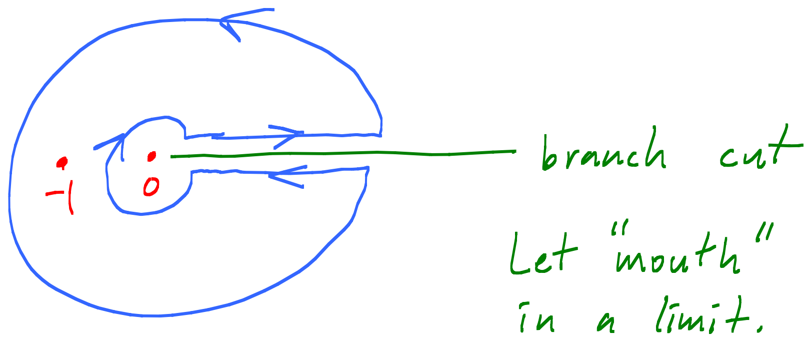
$$\Big(\underbrace{\int_{C_R}}_{\downarrow\, 0} + \underbrace{\int_{L_{Top}}}_{= \int_\varepsilon^R \frac{1}{\sqrt{t}(t+1)}dt} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\, 0} + \int_{L_{Bot}} \Big) f\, dz = 2\pi i \operatorname{Res}_{-1} f(z) = 2\pi$$

$=$

First let $R \to \infty$.
Then let $\varepsilon \to 0$

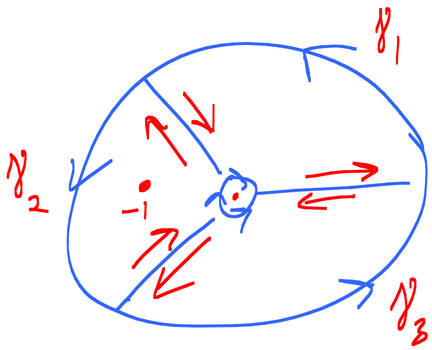
$2I = 2\pi$. So $I = \pi$.

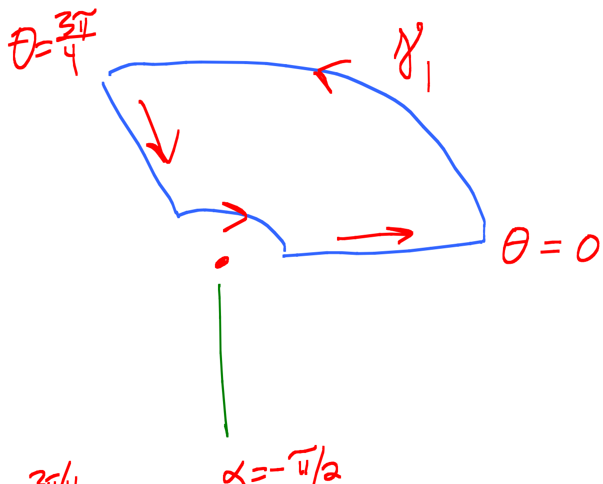
More carefully:

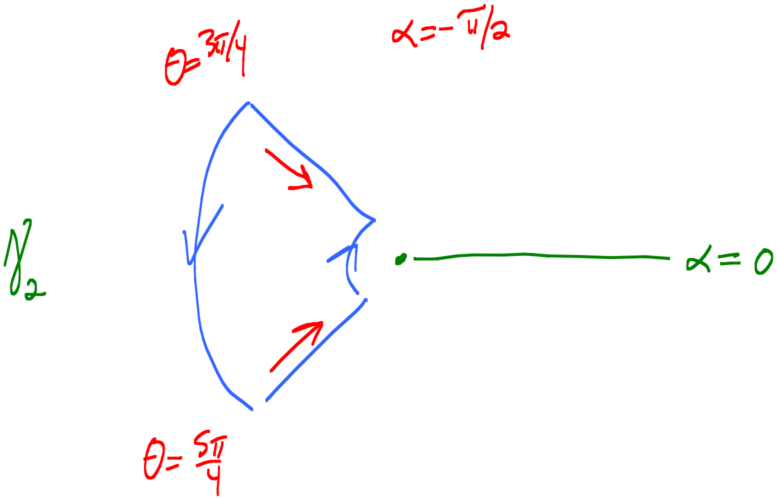


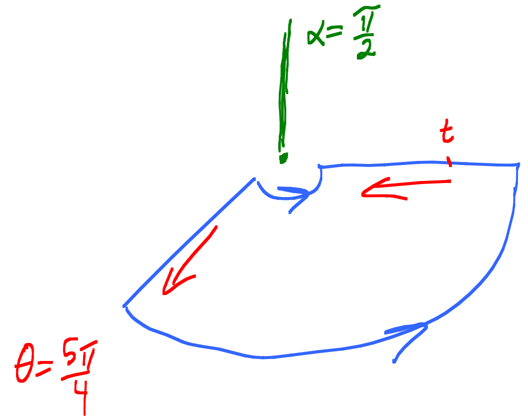
Let "mouth" close in a limit.

or









$\theta = 2\pi$!

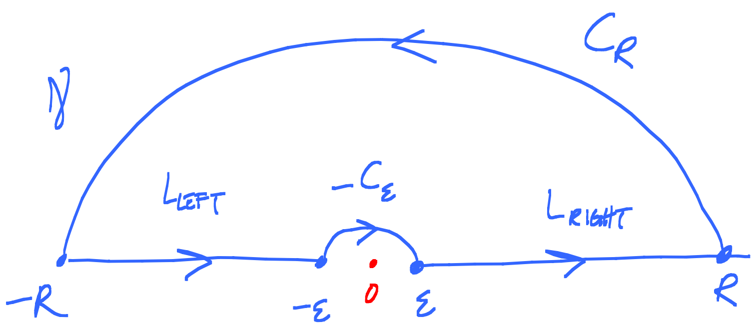
$$z^{1/2} = e^{\frac{1}{2}(\ln t + i\,2\pi)} = -\sqrt{t}$$

$z(t) = t$

EX: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

Need to use $\sin t = \operatorname{Im} e^{it}$

Let $f(z) = \dfrac{e^{iz}}{z}$



Cauchy's Thm says

$$\int_\gamma f\,dz = 0$$

$$\left(\int_{L_{LEFT}} + \int_{L_{RIGHT}}\right) \frac{e^{iz}}{z}\,dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \underbrace{\frac{e^{it}}{t}}_{\frac{\cos t}{t} + i\frac{\sin t}{t}} \cdot 1\,dt$$

$\frac{\cos t}{t}$: odd! $\qquad \frac{\sin t}{t}$: even

$$= 0 + i\,2\int_{\varepsilon}^{R} \frac{\sin t}{t}\,dt$$

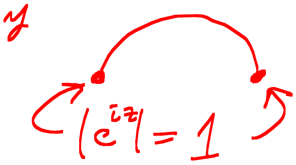
Big step: Show $\displaystyle\int_{C_R} f\,dz \to 0$ as $R \to \infty$. (Jordan's Lemma)

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \leq \left(\max_{C_R} \left|\frac{e^{iz}}{z}\right|\right) \cdot \pi R \leq \frac{1}{R} \cdot \pi R = \pi$$

$\leq$: Basic est.

π: ouch!

$$\max_{z \in C_R} \frac{|e^{iz}|}{|z|} = \frac{1}{R} \qquad \left| e^{i(x+iy)} \right| = \left| e^{ix} e^{-y} \right| = e^{-y}$$



Basic est fails us!

$$C_R : z(t) = Re^{it} \qquad 0 \le t \le \pi$$

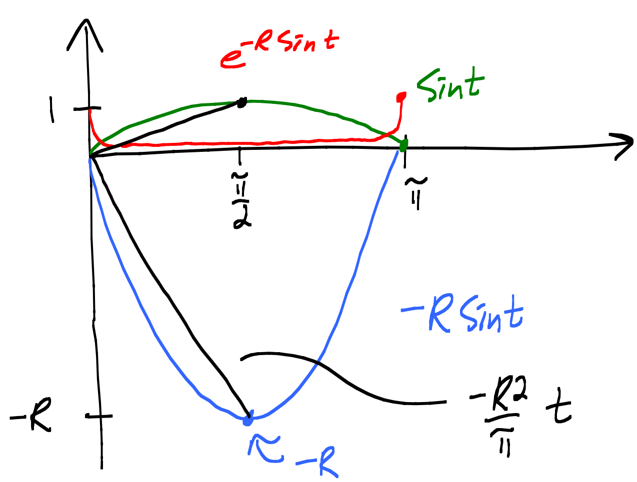
$$z'(t) = iRe^{it}$$

$$I_R = \left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| = \left| \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} \left(iRe^{it}\, dt \right) \right|$$

$$= \left| i \int_0^{\pi} \underbrace{e^{i(R\cos t + iR\sin t)}}_{e^{iR\cos t}\, e^{-R\sin t}} \, dt \right|$$

$e^{iR\cos t}$ ↑ modulus $= 1$

$$\le \int_0^{\pi} |\text{inside}|\, dt \qquad \int_0^{\pi} e^{-R\sin t}\, dt$$



$$\int_0^{\pi} e^{-R\sin t}\, dt$$

$$= 2 \int_0^{\pi/2} e^{-R\sin t}\, dt$$

↑ symmetry

$$< 2 \int_0^{\pi/2} e^{-\frac{R2}{\pi} t}\, dt$$

$$I_R < 2\left[-\frac{\pi}{R2}\, e^{-\frac{R2}{\pi}t}\right]_0^{\pi/2} = \frac{\pi}{R}\Big(\underbrace{1 - e^{-R}}_{<1}\Big) < \frac{\pi}{R}$$

$\to 0$ as $R \to \infty$!

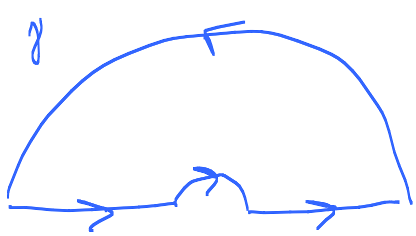
Last step: $\displaystyle\int_{-C_\varepsilon} f\,dz \to ?$

$$\frac{e^{iz}}{z} = \frac{1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \cdots}{z} = \frac{1}{z} + \underbrace{i - \frac{z}{2!} - \cdots}_{g(z)\ \text{entire!}}$$

$$\int_{-C_\varepsilon} f\,dz = -\int_{C_\varepsilon}\left[\frac{1}{z} + g(z)\right]dz$$

$$= \underbrace{-\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\left(i\varepsilon e^{it}\right)dt}_{-\int_0^{\pi} i\,dt \;=\; -i\pi} - \underbrace{\int_{C_\varepsilon} g\,dz}_{\to 0 \text{ as } \varepsilon \to 0}$$

$$\left|\int_{C_\varepsilon} g\,dz\right| \le \left(\max_{C_\varepsilon} |g|\right)(\pi\varepsilon) \to 0$$



$$0 = \int_\gamma f\,dz = \underbrace{\int_{C_R}}_{\downarrow\, 0} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\, -i\pi} + \underbrace{\left(\int_{\text{LEFT}} + \int_{\text{RIGHT}}\right)}_{\downarrow\, 2i\int_0^\infty \frac{\sin t}{t}\,dt}$$