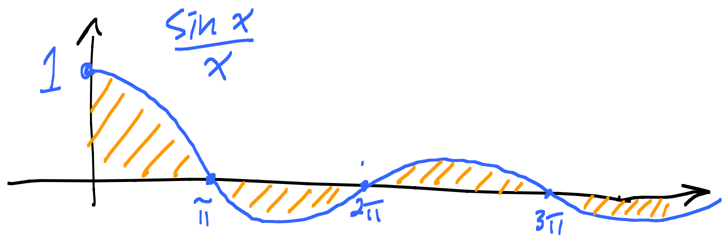
Lesson 29 Principal value integrals

28, 29, 30 due Wed.



$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Integral makes sense because of the Alternating Series Test

On $[n\pi, (n+1)\pi]$: $\quad \frac{|\sin x|}{(n+1)\pi} \le \frac{|\sin x|}{x} \le \frac{|\sin x|}{n\pi}$

Integrate $\int_{n\pi}^{(n+1)\pi}$: $\quad \frac{2}{(n+1)\pi} \le \left| \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx \right| \le \frac{2}{n\pi}$

area humps $\to 0$

Ouch $\quad \int_0^\infty \left|\frac{\sin x}{x}\right| dx \ge (\text{first hump}) + \underbrace{\sum_{n=1}^\infty \frac{2}{(n+1)\pi}}_{\infty} = \infty$

Famous example: $\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

Not absolutely convergent.

EX: $\int_0^\infty \frac{\cos x}{x^2+1}\,dx$ is absolutely conv.

$$\left|\frac{\cos x}{x^2+1}\right| \le \frac{1}{x^2+1} \qquad \int_0^\infty \left|\frac{\cos x}{x^2+1}\right| dx \le \int_0^\infty \frac{1}{x^2+1}\,dx$$

$$= \lim_{B\to\infty} \operatorname{Tan}^{-1} x \Big|_0^B$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} < \infty.$$

Good: Then $\displaystyle\int_{-\infty}^{\infty} = \underbrace{\lim_{R\to\infty} \int_{-R}^{R}}_{\text{Principal Value}} = \lim_{\substack{A\to-\infty \\ B\to+\infty}} \int_{A}^{B}$

Fact: For absolutely integrable fcns: Princ Val $=$ Value.

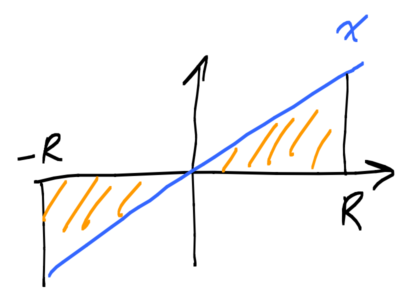
Ex: $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$

Big x: $\sim \dfrac{1}{x^{3/2}} \leftarrow$ int. as $x \to \infty$

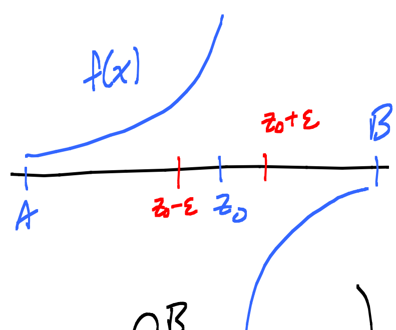
x near 0: $\sim \dfrac{1}{x^{1/2}} \leftarrow$ int on $(0,\varepsilon)$.

Ex: $\displaystyle PV \int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx$

$$= 0$$

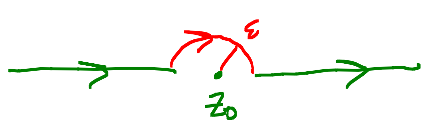


Another kind of PV integral:



$$PV \int_A^B f(x)\,dx = \lim_{\varepsilon\to 0} \left(\int_A^{z_0-\varepsilon} f\,dx + \int_{z_0+\varepsilon}^{B} f\,dx \right)$$

Why:



Two important facts:

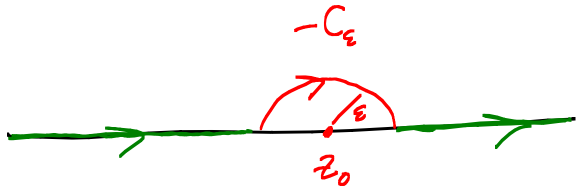
1) Jordan's Lemma yields

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \left(\sum_{\substack{z_K \text{ zeroes} \\ \text{of } Q \\ \text{in UHP}}} \operatorname{Res}_{z_K} \frac{P(z)}{Q(z)} e^{iz} \right)$$

when $\deg Q \geq \deg P + 1$ ← improved from $+2$ without Jordan's L.

and Q has no zeroes on $\mathbb{R}$.

2) General Fact: If f has a simple pole at $z_0 \in \mathbb{R}$:



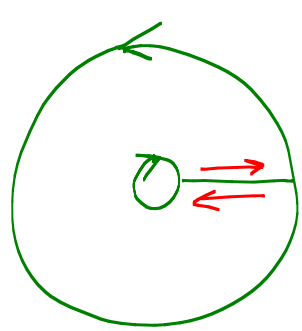
then $$\lim_{\varepsilon \to 0} \int_{-C_\varepsilon} f\, dz = -\frac{1}{2} \cdot 2\pi i \operatorname{Res}_{z_0} f$$

because clockwise

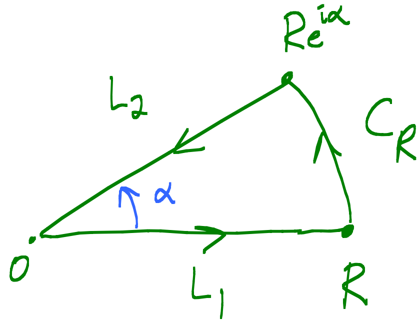
$\frac{1}{2}$ because it only goes half way around

Prob: $I = \displaystyle\int_0^\infty \frac{1}{x^5+1}\, dx$

$f(z) = \dfrac{1}{z^5+1}$



← ouch! They cancel!

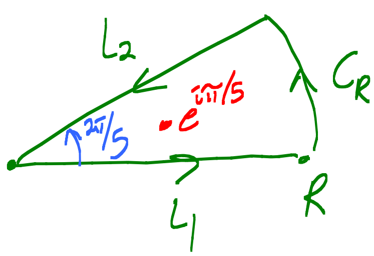


$$-L_2: \quad z(t) = t e^{i\alpha}, \quad 0 \le t \le R$$
$$z'(t) = e^{i\alpha}$$

Now $$\int_{L_2} f\,dz = -\int_{-L_2} f(z)\,dz = -\int_0^R \frac{1}{(te^{i\alpha})^5+1} \cdot \underbrace{e^{i\alpha}\,dt}_{\substack{z'(t)\,dt \\ = dz}}$$

Aha! Want $e^{i5\alpha} = 1$. $\alpha = \frac{2\pi}{5}$

Pole where $z^5 = -1$ inside: $z_0 = e^{i\pi/5}$



Simple pole at $e^{i\pi/5}$:

$$\frac{1 \leftarrow F}{z^5+1 \leftarrow G}$$

$G(e^{i\pi/5}) = 0$ ✓

$G'(z) = 5z^4$

$G'(e^{i\pi/5}) = 5e^{i4\pi/5}$ ← not zero

$$\operatorname{Res}_{e^{i\pi/5}} f = \frac{F(e^{i\pi/5})}{G'(e^{i\pi/5})} = \frac{1}{5(e^{i\pi/5})^4}$$

Step: $$\left|\int_{C_R} f\,dz\right| \le \left(\max_{C_R} \frac{1}{|z^5+1|}\right) \cdot \left(\frac{2\pi}{5} R\right) \underset{\substack{\text{if} \\ R>1}}{\le} \frac{1}{R^5-1} \cdot \frac{2\pi}{5} R \longrightarrow 0 \text{ as } R \to \infty.$$

Finally $$\left(\int_{L_1} - \int_{-L_2} + \int_{C_R}\right) f\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/5}} f$$

Let $R \to \infty$:

$$\int_0^R \frac{1}{t^5+1}\,dt - e^{i2\pi/5}\int_0^R \frac{1}{t^5+1}\,dt + \int_{C_R} f\,dz = \frac{2\pi i}{5e^{i4\pi/5}}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$I \quad - \quad e^{i2\pi/5}\, I \quad + \quad 0$$

So

$$I = \frac{2\pi i}{5\left(1-e^{i2\pi/5}\right)e^{i4\pi/5}} \cdot \frac{1}{e^{-i\pi/5}e^{i\pi/5}}$$

Real? ✓

$$= \frac{2\pi i}{5\left(e^{-i\pi/5}-e^{i\pi/5}\right)\underbrace{e^{i5\pi/5}}_{=-1}} = \frac{\pi}{5}\cdot\frac{1}{\sin\frac{\pi}{5}}$$

Fresnel integral: $\displaystyle\int_0^\infty \sin x^2\,dx$