Lesson 32 Conformal mapping

Exam 2 in class next Wed.

Harmonic conjugates: Know how to cook up on a disc:

Given harmonic u. Want v with $\begin{cases} v_x = -u_y & \text{(A)} \\ v_y = u_x & \text{(B)} \end{cases}$

$$\text{Want } \nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = \vec{F} \text{ and } \operatorname{curl} \vec{F} = \vec{0}.$$

$$\text{Get } v = \int_{\gamma_{z_0}^{z}} \vec{F} \cdot ds = \int_{\gamma_{z_0}^{z}} -u_y\,dx + u_x\,dy$$

or use $\int (A)\,dx$, $\frac{\partial}{\partial y}(\text{result}) = (B)$ to find v.

How to get v that works on simply connected Ω.

Given harmonic u on s.c. Ω.

Practice Prob: Show that $f(z) = u_x - i u_y$ is analytic on Ω. [Easy app of C-R Eqns.]

Hmmm. on a disc: If $F = u + iv$ is analytic, then

$$F' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} = u_x - i u_y$$

← analytic on disc and all of Ω.

Two Big Facts: 1) $\int_\gamma f\,dz = 0$ for all closed curves γ in a simply connected domain when f analytic.

2) $\int_\gamma f\,dz = 0$ for all closed γ in a domain (f analytic)

$\Longleftrightarrow \quad f = F'$ for some analytic F on Ω.

$$\left[\quad F(z) = \int_{\gamma_{z_0}^{z}} f \, dw \, . \right]$$

Consequently: Analytic fcns have analytic antiderivatives on simply conn. domains.

So our $f = u_x - i u_y$ has an anti-der $F = U + i V$.

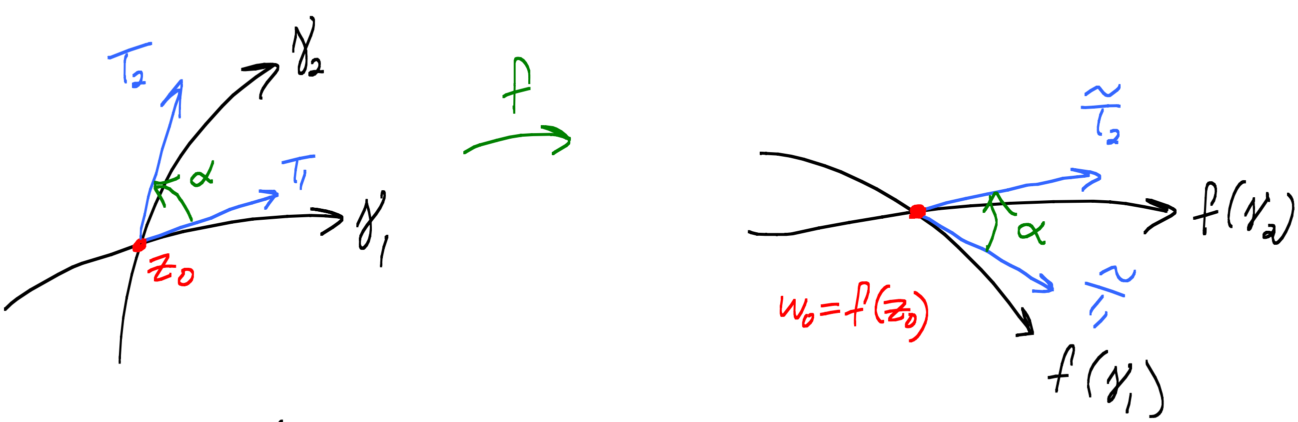
Claim: V is a harm conj for u on s.c. Ω.

Why: $F' = \begin{cases} U_x + i V_x \\ V_y - i U_y \end{cases} = U_x - i U_y = f = u_x - i u_y$

Aha! $\nabla U = \nabla u$. So $U = u + c$, $c =$ const. on Ω.

Now $\underset{\uparrow \text{ analytic}}{F} = (u+c) + iV = \underbrace{(u + iV)}_{\text{analytic.}} + c$ ✓

Today: Big fact: If f is analytic near z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0:



$\gamma_j : z_j(t)$, $\quad z_j(0) = z_0 \qquad T_j = z_j'(0) \qquad (j = 1, 2)$

$f(\gamma_j)$: $f(z_j(t))$, $f(z_j(0)) = f(z_0) = w_0$. $\tilde{T}_j = \frac{d}{dt} f(z_j(t)) \Big|_{t=0}$

Chain rule #2: $\tilde{T}_j = f'(z_j(t))\, z_j'(t) \Big|_{t=0}$

$$= \underbrace{f'(z_0)}_{re^{i\theta}}\, T_j$$

Aha! $\tilde{T}_j = \underbrace{(re^{i\theta})}\cdot T_j$ complex mult!

meaning: stretch by r, rotate by θ in c.c. direction.

So angle α and the sense are preserved.

Analytic f'cns (with $f' \neq 0$) are conformal.

Big fact: Reverse is true. (With care).

Suppose $(x,y) \mapsto (u(x,y), v(x,y))$ is C^1-smooth and the $\underbrace{\text{Jacobian det}}_{}$ is non-zero.

$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

If map preserves angles and sense, then u, v satisfy the CR Eqns [and so $f = u + iv$ is analytic]

Why: Soph calculus + analytic geom:

Jacobian matrix: $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ preserves angles if map conformal.

When does $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{R\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}} \overline{\overline{T}}$ preserve angles?

Aha! See the C-R Equs!

Riemann Mapping Theorem: If Ω is a simply connected domain and $\Omega \neq \mathbb{C}$, then there exists an analytic fcn $f: \Omega \to D_1(0)$ which is one-to-one and onto!

Know: f conformal, f' non-vanishing, f^{-1} is analytic too, so f^{-1} conformal too.

Prob from Exam 2 last year: Suppose f is analytic on $\mathbb{C} - \{a\}$ with a simple pole at $z = a$ [a pole of order 1], and f has a removable sing at ∞. Prove that

$$f(z) = \frac{A}{z-a} + B \quad \text{where } A, B \text{ are constants.}$$

Hmmm. Simple pole at a means $f(z) = \underbrace{\frac{A}{z-a}}_{\text{Princ. Part. near } a} + (\text{p.s.})$

So $f - \frac{A}{z-a}$ has a removable sing at a.

Let $F(z) = \begin{cases} f(z) - \frac{A}{z-a} & z \neq a \\ \text{proper value} & z = a \end{cases}$ to get F entire.

Hmmm. ∞ removable for f means $\lim_{z\to\infty} f(z)$ exists, say $=B$.

$$\text{Now} \quad \lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left(f(z) - \frac{A}{z-a}\right) = B - 0 = B.$$

Aha! F is bnd and entire. So $F \equiv \underbrace{\text{const.}}_{\text{must} = B}$

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