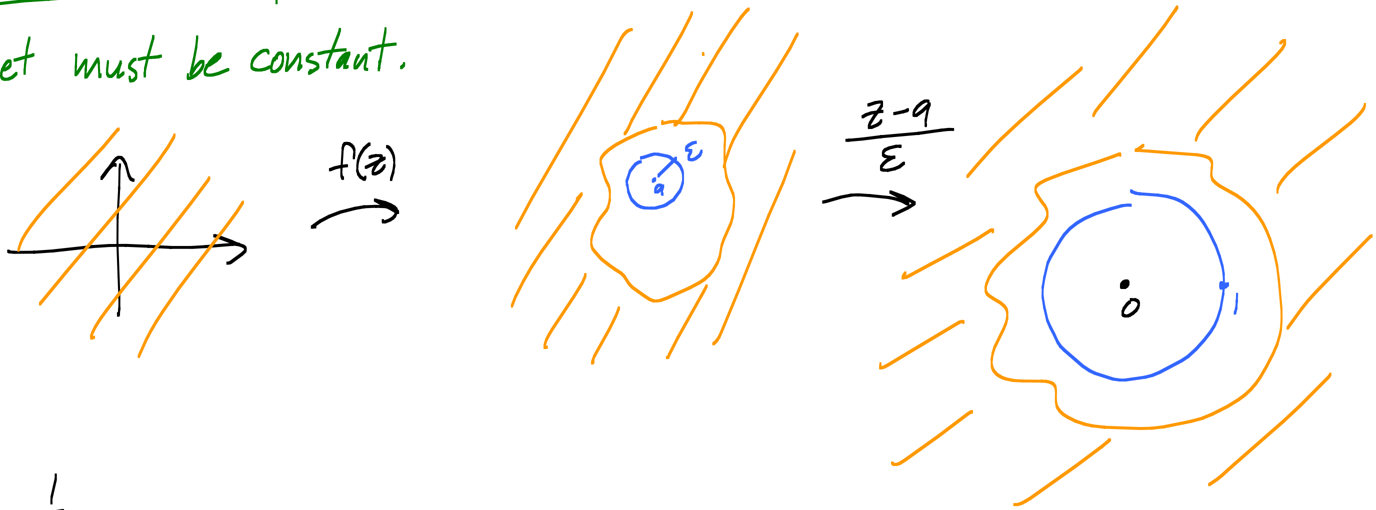
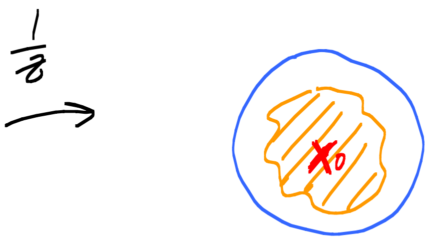
Lesson 40 The Schwarz Lemma   Practice material for Final Exam on home page

Casorati–Weierstraß Theorem: An entire function that misses an open set must be constant.

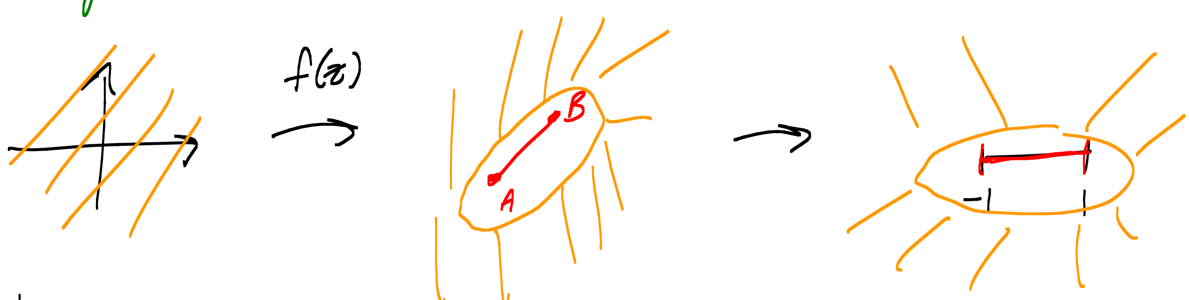


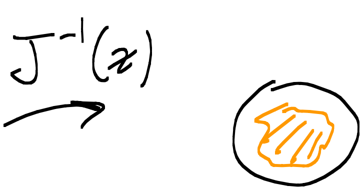


Aha! $\frac{\varepsilon}{f(z)-a}$ is a bdd entire fcn! Liouville's $\Rightarrow$

$\frac{\varepsilon}{f(z)-a} \equiv C$. So $f(z)$ is constant. ✓

How big a set can an entire fcn miss?





Missing a line segment $\Rightarrow f \equiv$ const.

Picard's Thm: An entire fcn that misses two points must be constant.

Same proof as for Casorati – W. yields this:

If f has an isolated sing at a and $f(D_\varepsilon(a) - \{a\})$ misses an open set, then a must be a pole or a removable sing.

Consequently, $f(D_\varepsilon(a)-\{a\})$ must be <u>dense</u> in $\mathbb{C}$ when a is ess.

<u>Schwarz Lemma</u>: Suppose f is analytic on $D_1(0)$ and

$f: D_1(0) \to D_1(0)$ $\quad [\, |f(z)| < 1$ when $|z| < 1 \,]$.

If $f(0) = 0$, then

A) $|f(z)| \le |z|$

B) $|f'(0)| \le \underline{1}$

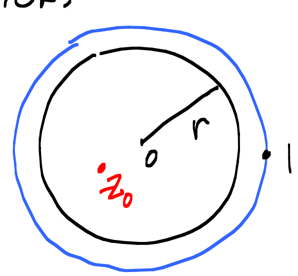
Furthermore, if equality holds for $z_0 \ne 0$ in (A), or if equality holds in (B), then $f(z) = \lambda z$ where λ is a const with $|\lambda| = 1$.

<u>Why</u>:

$$F(z) = \begin{cases} \dfrac{f(z)}{z} = \dfrac{f(z) - f(0)}{z - 0} & z \ne 0 \\ f'(0) & z = 0 \end{cases}$$

has a remov sing at $z = 0$

Trick: Think $0 < r < 1$, r close to 1.



$C_r(0)$

$|f(z)| < 1$

$$|F(z_0)| \le \max_{\overline{D_r(0)}} |F| = \max_{C_r(0)} |F| \le \frac{1}{r}$$

Now let $r \nearrow 1$. Get $|F(z_0)| \le 1$

$$\begin{cases} \left|\frac{f(z)}{z}\right| \le 1 & (A) \\ |f'(0)| \le 1 & (B) \end{cases}$$

Furthermore, if (A) $(z \neq 0)$ or (B) gives equality,

Max of $|F|$ happens at pt inside $D_r(0) \Rightarrow$

$F \equiv \lambda$, a const. $|\lambda| = 1$. $\quad \frac{f(z)}{z} = \lambda$

<u>Cor:</u> If $f: D_1(0) \to D_1(0)$ analytic and 1-1, onto and $f(0) = 0$, then $f(z) = \lambda z$ where λ is a const, $|\lambda| = 1$.

<u>Why:</u> Schwarz $\Rightarrow$ $|f'(0)| \le 1$. Apply Schwarz to f^{-1} too.

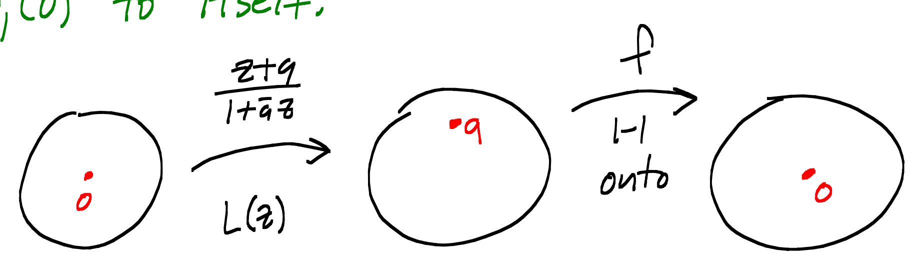
$$\left.\begin{aligned} |(f^{-1})'(0)| &\le 1 \\ \| \\ 1/f'(0) \end{aligned}\right\} \quad |f'(0)| \ge 1 \text{ too.}$$

Conclude $|f'(0)| = 1$. Furthermore of Schwarz $\Rightarrow f(z) = \lambda z$.

<u>Möbius Transformations:</u> $\lambda \cdot \dfrac{z - a}{1 - \bar{a} z}$ $\quad a \in D_1(0)$, $|\lambda| = 1$.

They form the family of all conformal one-to-one onto maps of $D_1(0)$ to itself.

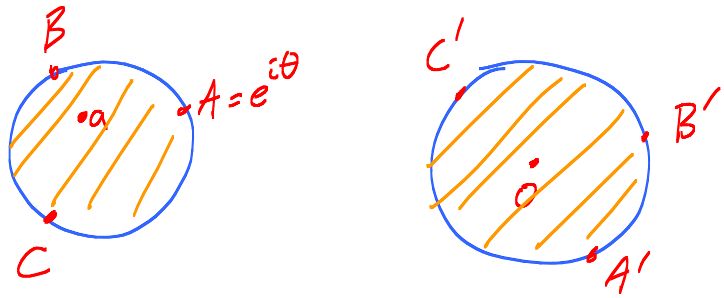
<u>Why:</u>



Cor to Schwarz $\Longrightarrow$ $f(\underbrace{L(z)}_{w}) = \lambda z$ $\quad z = L^{-1}(w)$

$$f(w) = \lambda L^{-1}(w) = \lambda \frac{w-a}{1-\bar{a}w} \checkmark$$

Missing fact: $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ maps $D_1(0) \to D_1(0)$ 1-1 onto.

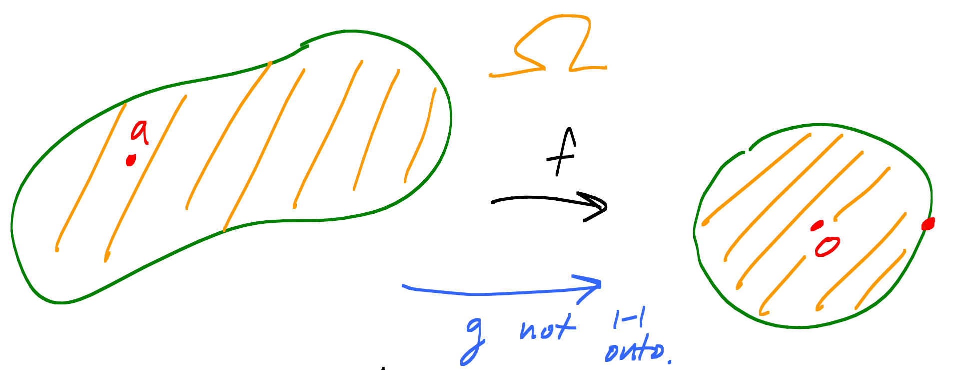


(One point test shows inside $\to$ inside.)

$$\varphi_a(e^{i\theta}) = \frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}} \cdot \frac{e^{-i\theta}}{e^{-i\theta}} = \frac{e^{-i\theta}\,(e^{i\theta}-a)}{(e^{-i\theta}-\bar{a})} = \frac{w}{\bar{w}}, \text{ modulus } = 1$$

So $|\varphi_a(e^{i\theta})| = 1$.

Riemann mapping thm:



The Riemann map maximizes $|f'(a)|$

$g \circ f^{-1} : D_1(0) \to D_1(0)$ and fixes 0.

$$|g'((f^{-1})(0))\,(f^{-1})'(0)| \le 1 \leftarrow |g'(a)| \le |f'(a)|$$