Review for Exam 1 ← in class on Wed.

Bring pencils, erasers.

1. Show that if $\varphi(x,y)$ is a twice continuously differentiable real valued *harmonic* function on a domain, then

$$U + iV = \underbrace{\left(\frac{\partial \varphi}{\partial x}\right)}_{U} - i\frac{\partial \varphi}{\partial y} \qquad V = -\frac{\partial \varphi}{\partial y}$$

is analytic there.

$U_x \overset{?}{=} V_y$ $\qquad$ $U_y \overset{?}{=} -V_x$

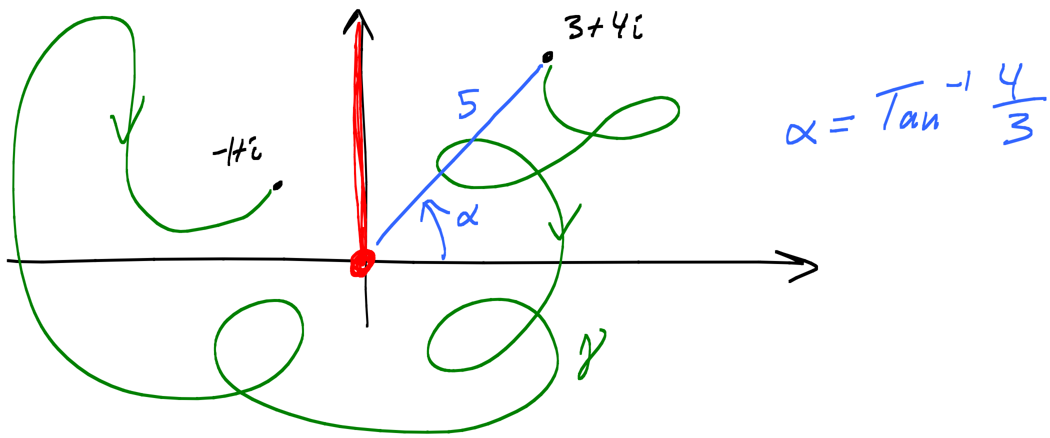
$\varphi_{xx} \overset{?}{=} -\varphi_{yy}$ ✓ $\qquad$ $\varphi_{yx} \overset{?}{=} -(-\varphi_{xy})$ ← yes, because φ is C^2-smth

U, V satisfy the C-R Eqns <u>and</u> are continuously diff'ble (because φ is C^2).

3. Compute

$$\int_\gamma \frac{1}{z}\,dz$$

where γ is any curve in the plane that starts at $3+4i$ and ends at $-1+i$ and that avoids the set $\{z : z = it, t \geq 0\}$ (i.e., the positive imaginary axis, including $z = 0$).



Define $\log_{\frac{\pi}{2}} z = \operatorname{Ln}|z| + i\theta$, $\theta \in \arg z$, $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$

Then $\displaystyle\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left(\log_{\frac{\pi}{2}} z\right) dz = \log_{\frac{\pi}{2}} z \Big|_{3+4i}^{-1+i}$

$$= \left(\operatorname{Ln}\sqrt{2} + i\,\frac{3\pi}{4}\right) - \left(\operatorname{Ln} 5 + i\,\boxed{2\pi + \operatorname{Tan}^{-1}\frac{4}{3}}\right)$$

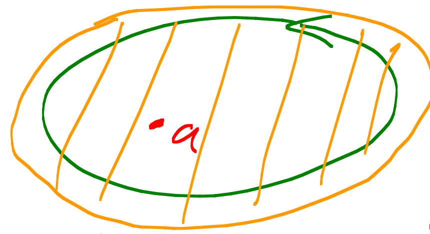
4. Define

$$I(a) = \int_C \frac{e^{5iz}}{(z-a)^4}\, dz, \qquad f(z) = e^{5iz}$$

where C is the unit circle parameterized in the counter clockwise direction and a is a complex number not on the unit circle. Compute $I(\frac{i}{3})$ and $I(3i)$. Is $I(a)$ an analytic function of a on the unit disc? Explain.

Know higher order Cauchy integral formulas and Cauchy Estimates.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$



γ cc simple closed

a inside γ

f analytic on [hatched region]

$n+1 = 4$. So $n = 3$

$$I(a) = \frac{2\pi i}{3!} f^{(3)}(a) \quad \text{where } f(z) = e^{i5z}$$

$$f'(z) = i5\, e^{i5z}$$

$$\vdots$$

$$f^{(3)}(z) = (i5)^3 e^{i5z}$$

if a inside C

$I(3i) = 0$ because of Cauchy's Thm.

5. Show that

$$\underbrace{|e^{(z^2)}|} \le \underbrace{e^{|z^2|}}_{e^{x^2+y^2}}$$

and identify conditions for equality to hold. Is it true that $|e^{(z^2)}|$ tends to infinity as $|z|$ tends to infinity?

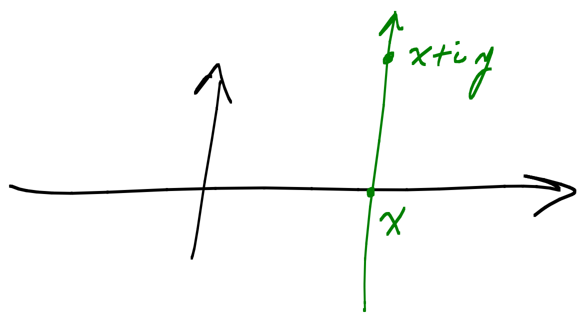
$z = x + iy$

$z^2 = (x^2 - y^2) + i2xy$

$$\underbrace{\left| e^{x^2-y^2} \cdot e^{i2xy} \right|}_{e^{x^2-y^2}} \overset{?}{\le} e^{x^2+y^2} \quad ✓$$

Only equal on $\mathbb{R}$.

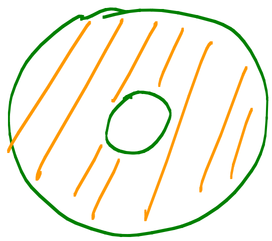
$|e^{z^2}| = e^{x^2-y^2} \to 0$ as $y \to \infty$ or $y \to -\infty$



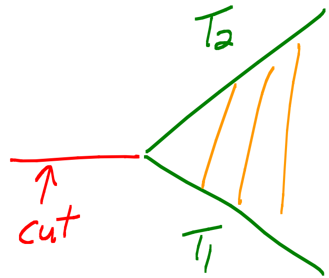
3. Find a continuous real valued function u on the annulus $\{z : 1 \leq |z| \leq 2\}$ that is harmonic inside the annulus, equal to 20 on the inner boundary and equal to 5 on the outer boundary.

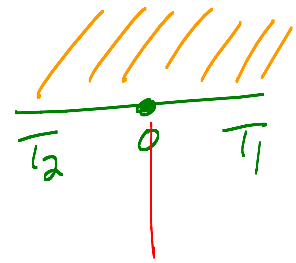
Key : $\ln|z| + i \arg z$ ← analytic branch of log

harmonic



$A \ln|z| + B$





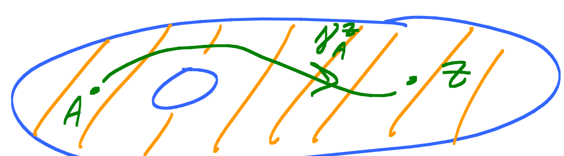
$A \arg z + B$

4. Use the Cauchy-Riemann equations to prove that a real valued analtyic function on a domain must be constant.

$f(x+iy) = u(x,y) + i\underbrace{v(x,y)}$

assume $v \equiv 0$

C-R eqns $\begin{cases} u_x = v_y \equiv 0 \\ u_y = -v_x \equiv 0 \end{cases}$



$u(z) - u(A) = \int_{\gamma_A^z} \nabla u \cdot d\vec{s} = 0$

So $\nabla u \equiv 0$ on a domain.

So $u \equiv$ const on Ω. ✓

9. Compute $\int_0^\pi e^{3it}\,dt$ where t is a real variable. $= \int_0^\pi z'(t)\,dt = z(\pi) - z(0)$ where $z(t) = \frac{1}{3i} e^{3it}$

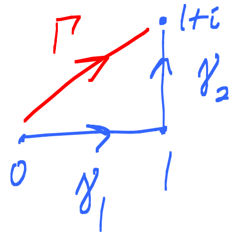
10. Compute the following path integrals

a) $\int_\gamma |z|^2\,dz$ where γ is the line from 0 to 1 followed by the line from 1 to $1+i$.

b) $\int_\Gamma |z|^2\,dz$ where Γ is the radial line from 0 to $1+i$.

11. Let γ denote any curve that starts at $2-i$ and ends at $-2-i$ and avoids the set $\{it : t \leq 0\}$. Compute a) $\displaystyle\int_\gamma \frac{1}{z^3}\,dz$ b) $\displaystyle\int_\gamma \frac{1}{z}\,dz$

10. $|z|^2$ not analytic!

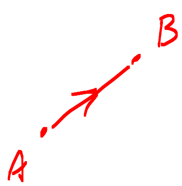


$\gamma_1: z(t) = t \quad 0 \le t \le 1$

$\gamma_2: z(t) = 1 + it, \; 0 \le t \le 1$

$z'(t) = i$

$$\int_{\gamma_2} |z|^2\,dz = \int_0^1 |\underbrace{1+it}_{z(t)}|^2 \,(\underbrace{i\,dt}_{z'(t)\,dt}) = i\int_0^1 (1+t^2)\,dt$$



$z(t) = A + (B-A)t, \quad 0 \le t \le 1$

11. a. $\frac{1}{z^3} = \frac{d}{dz}\left(-\frac{1}{2} \cdot \frac{1}{z^2}\right)$

z^{-3}

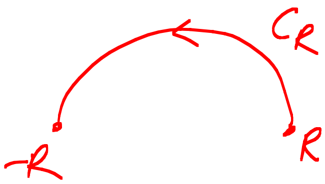
$\frac{1}{-3+1} \cdot z^{-3+1}$

12. Let C_R denote the half circle parameterized by $z(t) = Re^{it}$ for $0 \le t \le \pi$. Show that

Basic estimate:

$$\left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \le \left(\underset{C_R}{\text{Max}} \left| \frac{1}{z^4+1} \right| \right) \cdot \underbrace{(\pi R)}_{\text{length}(C_R)}$$

tends to zero as R tends to infinity.



Denominator estimate

$$|z-w| \ge \big| |z| - |w| \big|$$

$$|z+w| \ge \big| |z| - |w| \big|$$

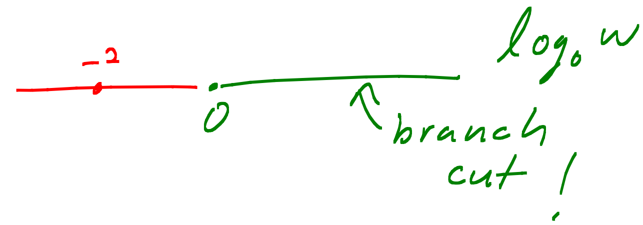
So $$|z^4+1| \ge \big| \underbrace{|z|^4}_{R^4} - 1 \big| = R^4 - 1 \quad \text{if } R > 1.$$

$$\left| \frac{1}{z^4+1} \right| \le \frac{1}{R^4-1} \quad \text{if } |z| = R > 1.$$

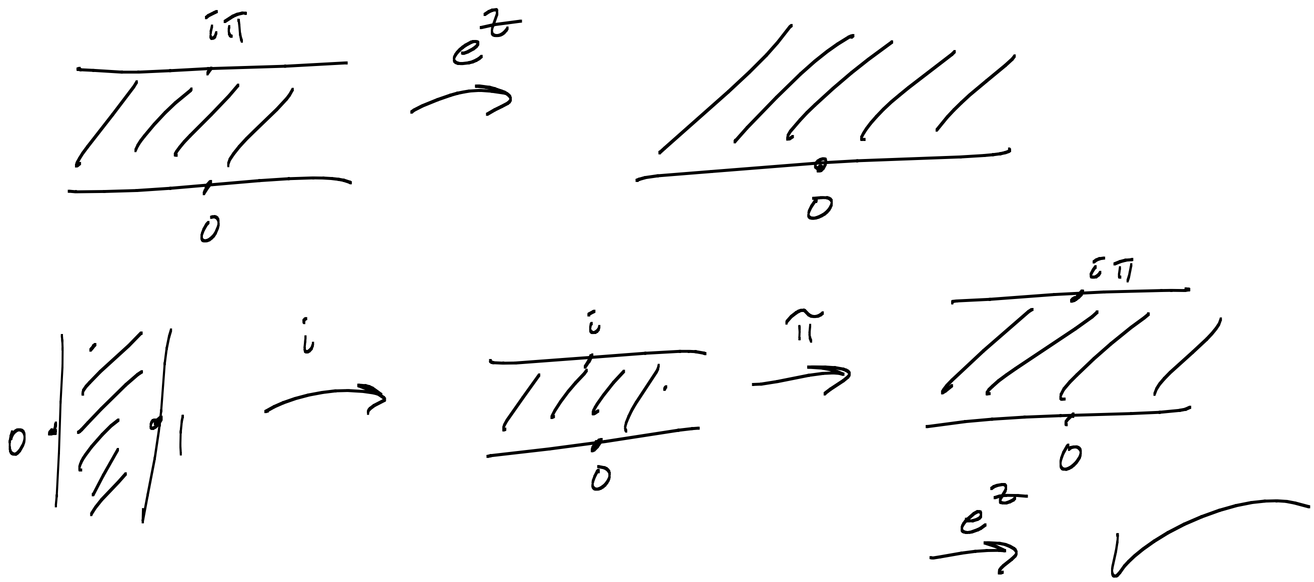
$$|I| \le \frac{1}{R^4-1} \cdot \pi R \longrightarrow 0 \quad \text{as } R \to \infty.$$

13. Determine a branch of $\log(\underbrace{z^2 + 4z + 1}_{=-2 \text{ when } z=-1.})$ that is analytic near $z = -1$ and find its derivative there.

14. Find a one-to-one analytic function that maps the strip $\{z : 0 < \operatorname{Re} z < 1\}$ onto the upper half plane.



14.



$$f(z) = e^{i\pi z}$$