Review for Exam 2 (in class Wed 11/14)

1. Let C denote the unit circle parameterized in the counterclockwise direction. Compute

$$\int_C \underbrace{\frac{e^{2z}}{2z^2+5z+2}}_{f(z)}\,dz. \quad = 2\pi i \sum_{\substack{a \text{ inside} \\ C}} \operatorname{Res}_a f(z)$$

(a = zeroes of denom)

Find roots of $p(z)$ = r_1, r_2. One in C, one out.

$p(r_j) = 0$, but $p'(r_j) \neq 0$. Simple zeroes!

$$\operatorname{Res}_{r_j} f(z) = \frac{e^{2r_j}}{p'(r_j)}$$

3. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{5n+7} z^{3n}.$$

u_n

$w = z^3$. w^n

Book: $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ ← Hate!

Ratio Test: $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{2^{n+1}}{5(n+1)+7} z^{3(n+1)}}{\frac{2^n}{5n+7} z^{3n}}\right|$

$$= 2 \cdot \frac{5n+7}{5n+12} |z|^3$$

$$\to 2|z|^3 \quad \begin{matrix} < 1 & \text{Conv} \\ > 1 & \text{div} \end{matrix}$$

$$R = \frac{1}{\sqrt[3]{2}}$$

What if I replace z^{3n} by z^{n^2}?

$$\left|\frac{u_{n+1}}{u_n}\right| = 2 \, \frac{5n+7}{5n+12} \left|\frac{z^{(n+1)^2}}{z^{n^2}}\right|$$

$$= 2 \, \frac{5n+7}{5n+12} |z|^{2n+1} \to \begin{matrix} 0 < 1 & \text{if } |z|<1 \\ \infty & \text{if } |z|>1 \end{matrix}$$

converges

diverges

So $R = 1$.

4. Calculate the residue at $z = 2i$ for the function

$$\frac{\text{Log } z}{(z^2+4)^2}, = \frac{\text{Log } z}{(z-2i)^2(z+2i)^2}$$

where Log z denotes the principal branch of the complex log function. Express your answer as a complex number $a + bi$ in simplest terms (i.e., evaluate things like Log $2i$ in your answer and simplify).

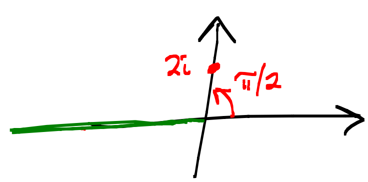
$$\frac{1}{(z-2i)^2}\left[\underbrace{\frac{\text{Log } z}{(z+2i)^2}}\right] \leftarrow \text{analytic near } 2i$$

$$H(z) = A_0 + A_1(z-2i) + \cdots$$

$$\frac{A_0}{(z-2i)^2} + \frac{A_1}{z-2i} + \text{power series.}$$

$$\text{Res}_{2i} = A_1 = \frac{H'(2i)}{1!} = \left.\frac{\left(\frac{1}{z}\right)(z+2i)^2 - (\text{Log } z)\,2(z+2i)}{(z+2i)^4}\right|_{z=2i}$$

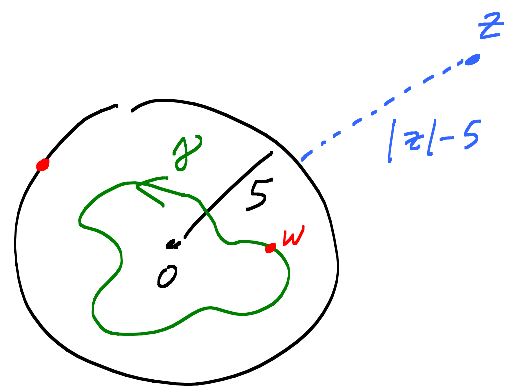
Note:



$$\text{Log } 2i = \text{Ln}|2i| + i\,\text{Arg } 2i$$
$$= \text{Ln } 2 + i\,\frac{\pi}{2}$$

5. Let γ denote a closed curve inside the disc of radius 5 about the origin and suppose that ϕ is a continuous complex valued function on γ. Let $F(z)$ denote the complex function defined for z on $\mathbb{C}$ minus the trace of γ given by

$$F(z) = \int_\gamma \frac{\phi(w)}{w - z}\, dw.$$



Use careful estimates to show that

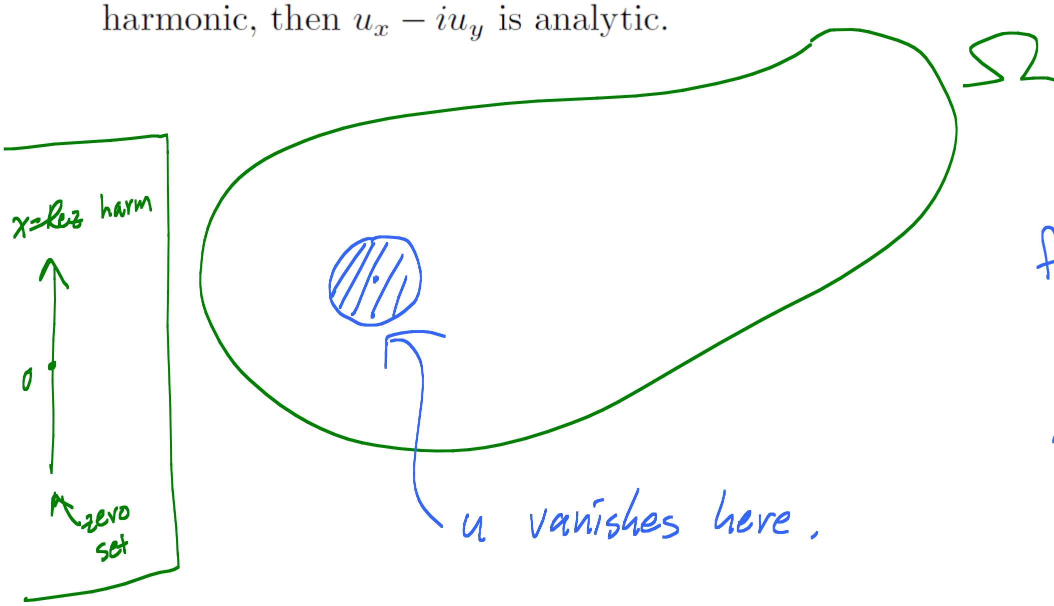
$$\lim_{z\to\infty} F(z) = 0.$$

$M = \underset{\gamma}{\text{Max}}\, |\varphi|$ exists.

$$|F(z)| \le \left(\underset{w \in tr(\gamma)}{\text{Max}} \left| \frac{\varphi(w)}{w-z} \right| \right) \text{Length}(\gamma)$$

$$\le \frac{M}{|z|-5} \cdot \text{Length}(\gamma) \quad \text{if } |z| > 5.$$

1. Prove that if a twice continuously differentiable harmonic function vanishes on an open subset of a domain, then it must be zero on the whole domain. Hint: If u is harmonic, then $u_x - iu_y$ is analytic.



u harmonic on Ω

$f = u_x - i u_y$ analytic on Ω.

Aha! f vanishes on $D_\varepsilon(a) \subset \Omega$.

limit pt of zero set.

Identity Thm $\Rightarrow f \equiv 0. \Rightarrow \nabla u \equiv 0 \Rightarrow u \equiv \text{const.}$ $(=0)$ ✓

2. Suppose that f and g are analytic on a disc about a and that each has a zero of multiplicity N at a. Prove a L'Hôpital's formula: $\displaystyle\lim_{z\to a}\frac{f(z)}{g(z)} = \frac{f^{(N)}(a)}{g^{(N)}(a)}$.

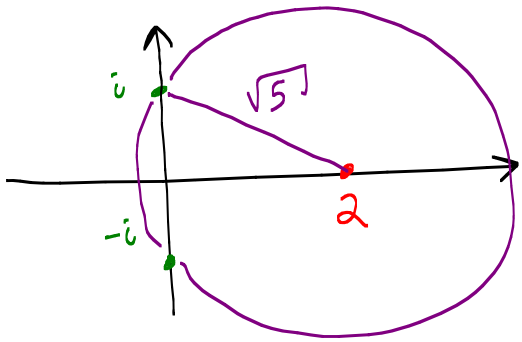
$$f(z) = a_N (z-a)^N + \cdots = (z-a)^N \Big[\underbrace{a_N + a_{N+1}(z-a) + \cdots}_{F(z)}\Big]$$

$$g(z) = b_N (z-a)^N + \cdots = (z-a)^N \Big[\underbrace{b_N + \cdots}_{G(z)}\Big]$$

$$\frac{f(z)}{g(z)} = \frac{F(z)}{G(z)} \xrightarrow[z\to a]{} \frac{F(a)}{G(a)} = \frac{a_N}{b_N} = \frac{\left(\frac{f^{(N)}(a)}{N!}\right)}{\left(\frac{g^{(N)}(a)}{N!}\right)} \checkmark$$

3. Let $f(z) = \frac{1}{1+z^2}$. Find the first three terms in the power series $\sum_{n=0}^{\infty} a_n(z-2)^n$ for f centered at $z = 2$. What is the radius of convergence of the series? Explain.

Use $\quad a_n = \dfrac{f^{(n)}(2)}{n!} \quad n = 0, 1, 2$

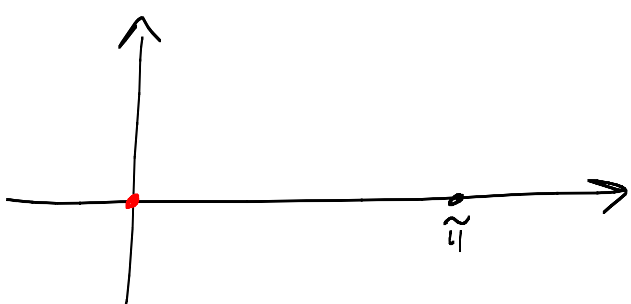


f analytic on $D_{\sqrt{5}}(2)$.

So $R \geq \sqrt{5}$.

f blows up at $\pm i$. So $R \leq \sqrt{5}$ too. $\quad R = \sqrt{5}$.

4. What is the radius of convergence of the power series for $\frac{\sin z}{z}$ centered at $z = \pi$?



$$F(z) = \begin{cases} \overset{z \neq 0}{\dfrac{\sin z}{z}} & \text{has a } \underline{\text{removable}} \text{ sing at } 0 \\ 1 & z = 0 \end{cases}$$

is $\underline{\text{entire}}$.

$\dfrac{\operatorname{Log} z}{z-1}$

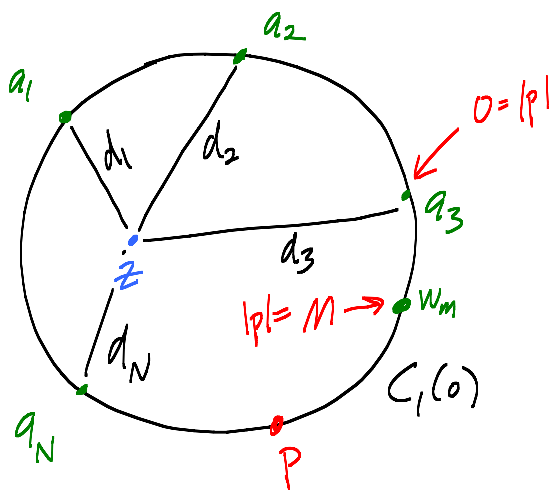
$a = 2$

$R = 2$, not 1.

Power series for f same as for F.

So $R = \infty$.

6. Suppose $a_1, a_2, \ldots, a_n$ are points on the unit circle. For a point z on the unit circle, let $D(z)$ denote the product of the distances from z to each of the n points on the circle. Prove that there is a point z on the circle such that the product $D(z)$ is exactly equal to one. Hint: Let $p(z)$ denote the product of $(z - a_n)$ as n ranges from one to n. What is $|p(0)|$? Use the Maximum Modulus Principle.



$$D = \prod_{k=1}^{N} d_k$$

$D(p) = 1$ exists.

$$p(z) = \prod_{k=1}^{N} (z - a_k)$$

$|p(z)| = D(z)$ $\quad$ p poly of deg $N > 1$.

p is not const.

$|p(0)| = \prod \underbrace{|a_k|}_{1} = 1 \leftarrow$ can't be max modulus (otherwise $p \equiv$ const. ↯)

Max Modulus Thm $\Rightarrow \exists\ w_m \in C_1(0)$ where $|p(w_m)| = \underset{\substack{\uparrow \\ \text{max modulus.}}}{M} > 1$

$|p(a_k)| = 0$, $|p(w_m)| = M > 1$

Intermediate value thm.

$\Rightarrow \exists$ place between a_k and w_m on $C_1(0)$

where $|p| = 1$, ✓

9. Assume $b_0 = 1$. Constants $b_2, b_4, b_6, \ldots$ are generated via the recursion formula

$$b_{n+2} = \frac{4(n+3)(n-5)}{7(n+1)(n+2)}\, b_n.$$

What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} b_{2n} z^{2n} = b_0 + b_2 z^2 + b_4 z^4 + \cdots ?$$

$$\lim_{\text{out} \to \text{way out}} \left| \frac{u_{\text{out there}}}{u_{\text{before}}} \right|$$

and use recursion formula

10. If the complex numbers a_n tend to zero as n tends to infinity, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least one.

Hadamard's: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \sup_{n\geq N} \sqrt[n]{|a_n|}$

$0 \leq |a_n| < 1$ if $n > N_0$ since $a_n \to 0$.

Now take n-th root: $0 \leq \sqrt[n]{|a_n|} < 1$. Aha! Lim of Sups has to be ≤ 1.

So $\frac{1}{R} \leq 1$.

And $R \geq 1$. ✓

Or better yet, since $a_n \to 0$, it is a bounded seq.

So $\exists\, M$ such that $|a_n| \leq M$ for all n.

Now, if $|z| < 1$, then $|a_n z^n| = |a_n||z|^n \leq \underbrace{M|z|^n}_{\text{Conv. Geom Series}}$

Comparison test shows $\sum a_n z^n$ conv if $|z| < 1$. So $R \geq 1$.