Review 2

Final Exam Wednesday, December 12, 3:30-5:30 pm in BRNG 2290

Office Hours Mon, Tues 2:00–3:30

1. Suppose f and g are two entire functions such that g is not identically equal to zero and

$$|f(z)| \leq |g(z)|$$

for all z. Prove that $f = cg$ for some constant c with $|c| \leq 1$. (Be mindful of points where g might vanish.)

Hmmm. $F(z) = \frac{f(z)}{g(z)}$. Then $|F(z)| \leq 1$ ... where $g(z) \neq 0$.

Case $g \equiv$ const: Liouville's $\Rightarrow$ f const. too. ✓

Bdd entire fcn must be const.

Case $g \not\equiv$ const: Then zeroes of g must be isolated.

Suppose $g(z_0) = 0$. Then $\exists\ \varepsilon > 0$ such that z_0 is the only zero of g in $D_\varepsilon(z_0)$. Now $\frac{f(z)}{g(z)}$ has an isolated sing at z_0. $\left|\frac{f}{g}\right| < 1$ on $D_\varepsilon(z_0) - \{z_0\}$.

R.R.S. Thm $\Rightarrow$ z_0 is removable.

Aha! $F(z) = \begin{cases} \frac{f(z)}{g(z)} & g(z) \neq 0 \\ \text{proper value} & \text{at places where } g = 0. \end{cases}$

is a bnd entire fcn. So $F \equiv c$. $\frac{f}{g} \equiv c$

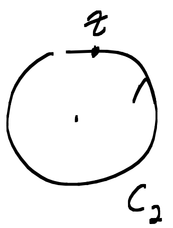
So $f = cg$. $\leftarrow$ holds at places wher $g(z) = 0$ too by continuity.

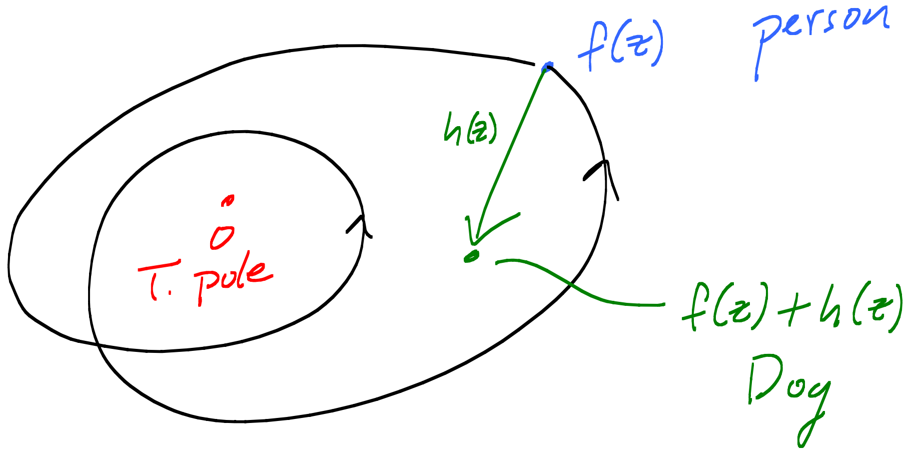
4 zeroes counted with mult.

6. How many zeroes does the polynomial $z^4 + z^3 - z^2 + 2$ have inside the disc of radius two about the origin? Explain.

$16 \quad 8 \quad 4 \quad 2$

Rouché's!





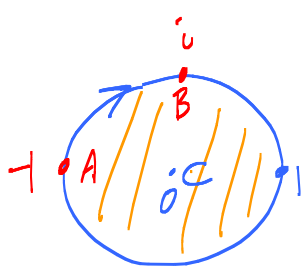
$\underbrace{|h(z)|}_{\text{leash}} \le |f(z)| \Rightarrow f$ and $f+h$ have same # zeroes in $C_2(0)$.

Let $f(z)=z^4$ and $h(z)=z^3-z^2+2$. on $C_2(0)$:

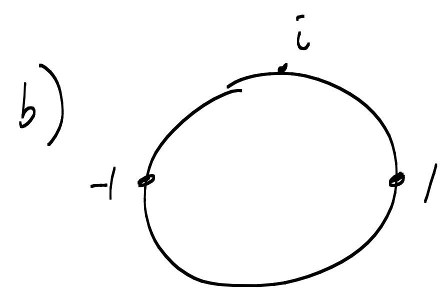
$|h(z)| \le |z^3| + |z^2| + 2 \le 8+4+2 = 14 < 16 = |z^4| = |f(z)|$.

7. What is the image of the unit disc under the following mappings?

a) $T_1(z) = z/(1-z)$ b) $T_2(z) = z/(z - i/2)$

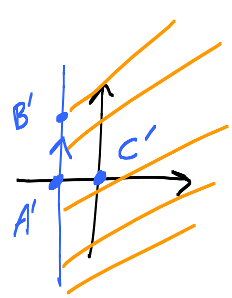


b)



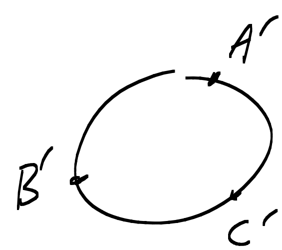
a) boom! at $z=1$.

unit circle $\rightarrow$ line

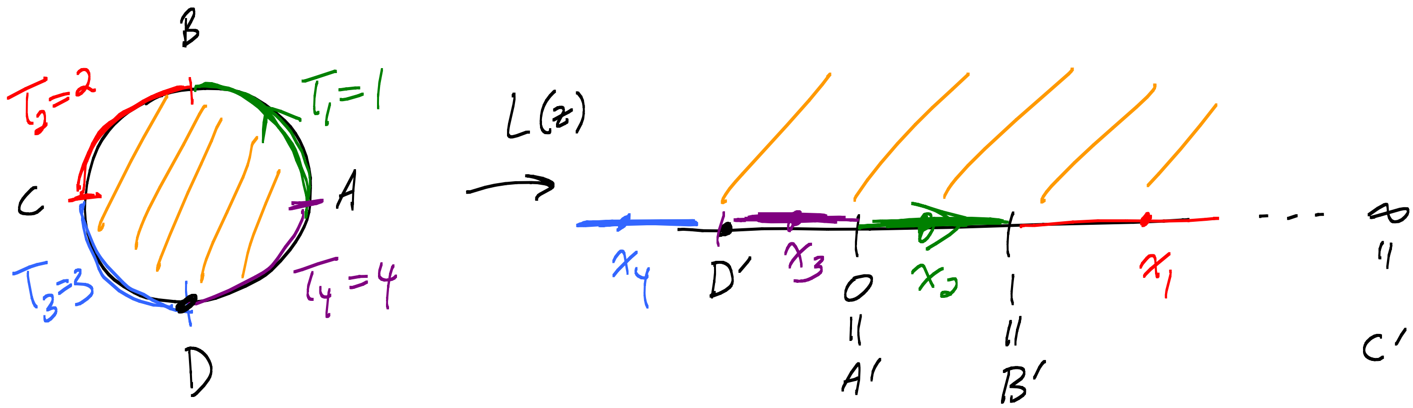


$T_1(-1) = \frac{-1}{1-(-1)} = -\frac{1}{2}$

$T_1(i) = \frac{i}{1-i} \cdot \frac{1+i}{1+i} = \frac{-1+i}{2} = -\frac{1}{2} + i\frac{1}{2}$



8. Find a harmonic function on the unit disc that has boundary values on the unit circle equal to one in the first quadrant, two in the second quadrant, three in the third quadrant, and four in the fourth quadrant.



$$L(z) = \frac{B-C}{B-A}\,\frac{z-A}{z-C} = \frac{i-(-1)}{i-1}\cdot\frac{z-1}{z-(-1)}$$

$$u(z) = a\,\mathrm{Arg}(z-D') + b\,\mathrm{Arg}(z-0) + c\,\mathrm{Arg}(z-1) + d$$

At x_1: $0+0+0+d = 2$

At x_2: $0+0+c\cdot\pi+d = 3$

$\vdots$

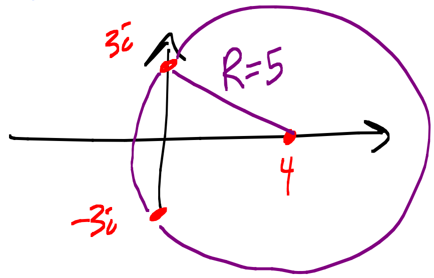
Sol^n: $u(L(z))$

4. What is the radius of convergence of the power series for $\dfrac{\sin z}{z^2+9}$ centered at $z = 4$? Explain.

← entire

← zeroes at $z = \pm 3i$
$\sin(3i) \neq 0.$

$f(z)$ has poles at $\pm 3i$



f analytic on $D_5(4)$.
So $R \geq 5$.

f blows up at $\pm 3i$. So $R \leq 5$. $\underline{R=5}$

3. Let C_1 denote the unit circle parametrized in the counterclockwise sense. Evaluate the following integrals. Explain.

a) $\displaystyle\int_{C_1} \frac{e^{2z}}{z-4}\,dz$ $= 0$ by Cauchy's Thm (not $2\pi i\, e^{2\cdot 4}$!)

b) $\displaystyle\int_{C_1} \frac{e^{2z}}{4z^2+1}\,dz$ $= 2\pi i \left(\underbrace{\operatorname{Res}_{\frac{i}{2}} \frac{e^{2z}}{4z^2+1}} + \operatorname{Res}_{-\frac{i}{2}} \cdots \right)$

$= \left.\dfrac{e^{2z}}{8z}\right]_{z=i/2}$ because $\frac{i}{2}$ is a $\underline{\text{simple}}$ zero of denom

c) $\displaystyle\int_{C_1} \frac{e^{2z}}{2z^2-5z+2}\,dz$

d) $\displaystyle\int_{C_1} \frac{e^{2z}}{(z-1/2)^5}\,dz$

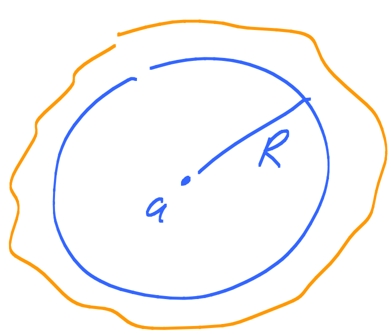
$\dfrac{1}{(z-\frac{1}{2})^5}\left[A_0 + A_1(z-\tfrac{1}{2}) + \cdots + A_4(z-\tfrac{1}{2})^4 + \cdots\right]$

$\operatorname{Res} = \left.\dfrac{1}{4!}\dfrac{d^4}{dz^4} e^{2z}\right|_{z=\frac{1}{2}}$

e) $\displaystyle\int_{C_1} \frac{1}{z^5}\,dz$ $= 0$

because $\dfrac{1}{z^5} = \dfrac{d}{dz}\left[\dfrac{-1/4}{z^4}\right]$ or Res. Thm.

$\underline{\text{Cauchy Estimates}}$



f analytic

$$|f^{(n)}(a)| \leq \frac{n!\,M}{R^n} \qquad M = \max_{C_R(a)} |f|$$