MA 425 / 525 Complex variables

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$x^2+1=0$ Ouch! Toss in $\sqrt{-1}$. Call it i. $\sqrt{-1}=\pm i$

Also need $3+i$, $\pi+ei$, ..., $(3+4i)^{10}=\dots=a+bi$

$\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}$

$$(a+bi)^2=a^2+2abi+\underbrace{(bi)^2}_{-b^2}=(a^2-b^2)+(2ab)i$$

$$z_1+z_2=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+(y_1+y_2)i$$

$$z_1\cdot z_2=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i$$

Zero element : $0+0i \leftarrow$ write 0

Identity elem : $1+0i \leftarrow$ write 1

Big fact: $\mathbb{C}$ is a "field"

Multiplicative inverse : $z^{-1}=\dfrac{1}{z}=\dfrac{1}{x+iy}\cdot\dfrac{x-iy}{x-iy}=\dfrac{x-iy}{x^2+y^2}$

$$=\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}\,i$$

Hmmm: Want w with $z\cdot w=1$

$$(x+iy)(u+iv)\overset{\text{want}}{=}1$$

$$(ux-vy)+(vx+uy)i=1$$

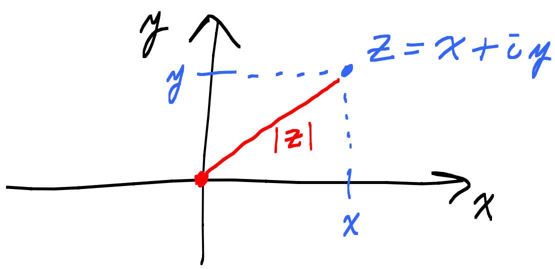
$\underbrace{\qquad}_{1} \qquad \underbrace{\qquad}_{0}$

Cramer's rule: $x = \dfrac{\det\begin{pmatrix} 1 & -v \\ 0 & u \end{pmatrix}}{\det\begin{pmatrix} u & -v \\ v & u \end{pmatrix}} = \dfrac{u}{u^2+v^2}$ ✓

$y = \ldots$

Inverse is unique!

Notation:



$|z| = \sqrt{x^2+y^2}$

= "modulus of z"

• $\bar{z} = x - iy \leftarrow$ complex conjugate of z

Facts: $\overline{(zw)} = \bar{z}\,\bar{w}$

$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

$z \cdot \bar{z} = |z|^2$

Notice: $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

Quadratic Eqns:

$x^2 + bx + c = 0$

$\left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) = 0$

$\left(x + \frac{b}{2}\right) = \pm\sqrt{\underbrace{\left(\frac{b^2}{4} - c\right)}_{}}$

Ouch if this

$b^2 - 4ac < 0$

Fundamental Theorem of Algebra: Any complex polynomial

$$P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$$

$a_N \neq 0$

of degree $N \geq 1$ has a complex root!

[$\exists\, z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.]

Cor: $P(z)$ factors over $\mathbb{C}$!

$$P(z) = a_N (z - z_1)(z - z_2) \cdots (z - z_N)$$

(some z_j's might be the same.)

Exciting thing: Calculus with complex #'s.

$$DQ = \frac{f(z) - f(a)}{z - a} \xrightarrow[z \to a]{} L, \text{ a limit.}$$

Say $L = f'(a)$ ← complex derivative

EX: $f(z) = z^2$

$$DQ = \frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z-a} = z + a \xrightarrow[\substack{\text{as} \\ z \to a}]{} a + a = 2a$$

$f'(a) = 2a$. And $f'(z) = 2z$

EX: $f(z) = \bar{z} = x - iy$ ← not complex diff'ble!

Complex exponential:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

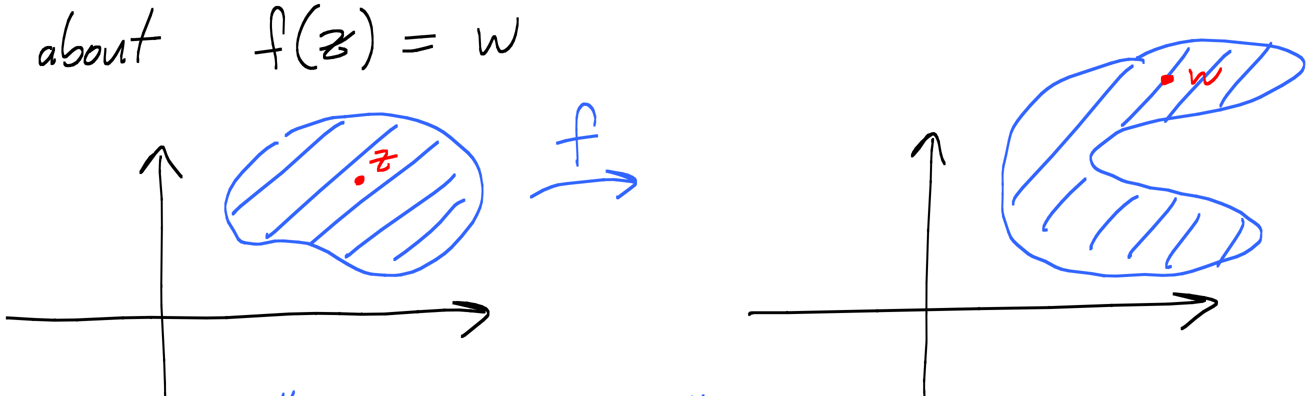
$$= e^{x+iy} = e^x e^{iy} \quad \leftarrow \text{Euler}$$

$$= e^x \left[1 + (iy) + \frac{(iy)^2}{2!} + \cdots \right]$$

$$= e^x \left[\underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}_{\cos y} + i \underbrace{\left(y - \frac{y^3}{3!} + \cdots\right)}_{\sin y} \right]$$

$$\boxed{e^{x+iy} = e^x(\cos y + i \sin y)} \leftarrow \text{def}^n \text{ of } e^z$$

Feel about $f(z) = w$



"Sheet of rubber"