Lesson 2 Complex numbers, complex exponential

$$|zw| = |(x+iy)\cdot(u+iv)| = |(xu-yv)+i(xv+yu)|$$

$$= \sqrt{(xu-yv)^2+(xv+yu)^2} = \dots = \sqrt{x^2+y^2}\cdot\sqrt{u^2+v^2} = |z|\,|w|$$

Def^n: $e^z = e^{x+iy} = \left(e^x \cos y\right) + i\left(e^x \sin y\right)$

Properties:

1) $e^0 = 1$
2) $\frac{d}{dz}\left(e^z\right) = e^z$

$\}$ determine e^z !

3) $e^{z+w} = e^z e^w$
4) $e^{-z} = 1/e^z$ $\leftarrow$ so e^z is never zero

DeMoivre's Formula: $\left(e^{i\theta}\right)^N = e^{iN\theta}$

why: $(e^{i\theta})^N = \underbrace{e^{i\theta}\cdot e^{i\theta}\cdot\dots\cdot e^{i\theta}}_{N\text{- times}} = e^{i\theta+i\theta+\dots+i\theta} = e^{iN\theta}$

Check today: $e^{i(\alpha+\beta)} = e^{i\alpha}\cdot e^{i\beta}$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \left[\cos\alpha\cos\beta - \sin\alpha\sin\beta\right] + i\left[\cos\alpha\sin\beta + \sin\alpha\cos\beta\right]$$

$=$ $=$

Pf of (3): $e^z e^w = e^{x+iy}e^{u+iv} = e^x e^{iy} e^u e^{iv}$

$$= \underbrace{\left(e^{x} e^{u}\right)}_{e^{x+u}} \underbrace{\left(e^{iy} e^{iv}\right)}_{e^{i(y+v)}} = e^{z+w} \checkmark$$

Cool thing: Get trig identities via algebra!

EX: $\sin 2\theta = 2\sin\theta\cos\theta$

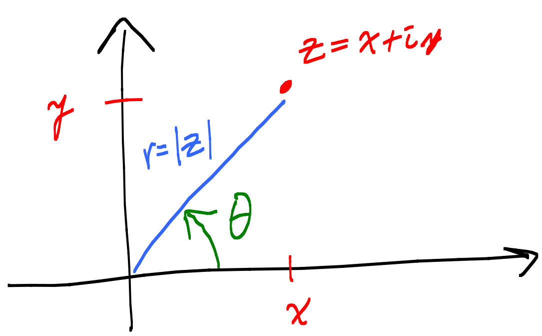
Why? $\underbrace{e^{i2\theta}}_{\cos 2\theta + i\sin 2\theta} = e^{i\theta}\cdot e^{i\theta} = \cdots = \ell\ell\ell + i\underbrace{\left(2\sin\theta\cos\theta\right)}$

$=$

Arithmetic on $\mathbb{C}$: Algebra – just like $\mathbb{R}$

Analysis on $\mathbb{C}$: Topology just like $\mathbb{R}^2$

Complex #'s in polar form via e^z:



$x = \operatorname{Re} z$

$y = \operatorname{Im} z$

$|z| = \sqrt{x^2+y^2}$ = modulus of z

$\theta = \underline{\operatorname{Arg}} z$ = Principal argument of z

$= \theta$ in picture with $-\pi < \theta \le \pi$

$\arg z = \{\theta = \operatorname{Arg} z + 2\pi n \ : \ n \in \mathbb{Z}\}$

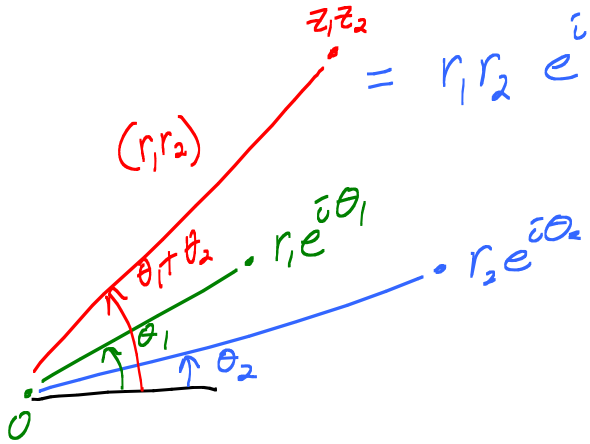
$\{0, \pm 1, \pm 2, \ldots\}$

Aha! $z = x + iy = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta) = \underline{\underline{r e^{i\theta}}}$

Great for multiplication $\quad z_1 \cdot z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$

$$= r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$



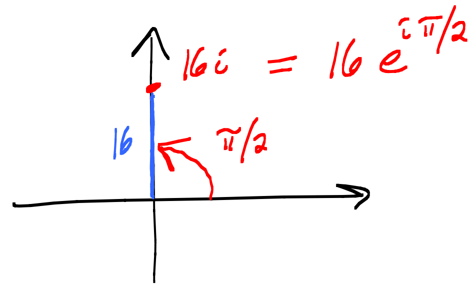
Remarks: $\quad \left| e^{i\theta} \right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \overline{e^{i\theta}}$$

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

Finding complex roots: $\quad$ Ex: Find all z such that

$$z^4 = 16i$$



$$\left(re^{i\theta}\right)^4 = 16\, e^{i\pi/2}$$

$$r^4 e^{i4\theta} = 16\, e^{i\pi/2}$$

want

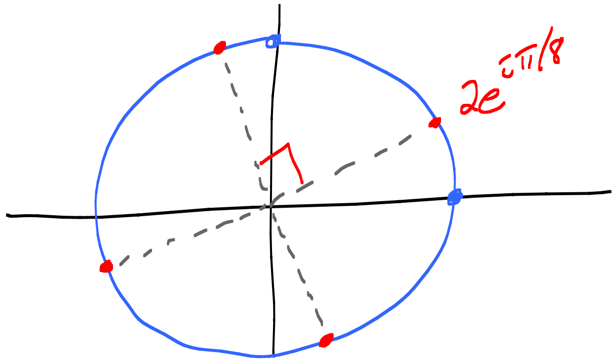
Need $\boxed{r^4 = 16}$

$\underline{\underline{r = 2}}$

Also need $\quad 4\theta = \frac{\pi}{2} + 2\pi n$

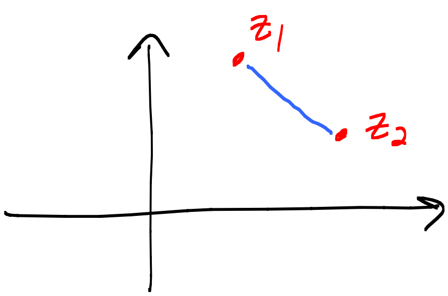
So $\quad \underline{\underline{\theta = \frac{\pi}{8} + n\frac{\pi}{2}, \quad n \in \mathbb{Z}}}$

Ans: $\quad z = 2e^{i\left(\frac{\pi}{8} + n\frac{\pi}{2}\right)} \qquad n = 0, \pm 1, \pm 2, \ldots$



Only 4 !

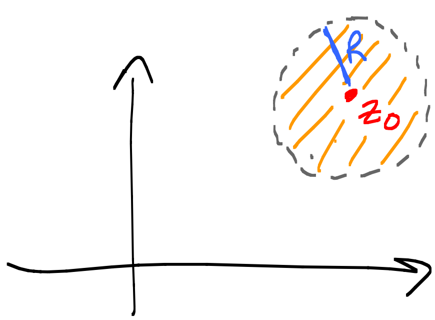
Topology of $\mathbb{C}$: Distance function on $\mathbb{C} \approx \mathbb{R}^2$



$$\text{dist}(z_1, z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

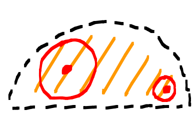
$$= |z_1 - z_2|$$

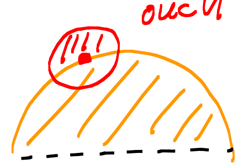
$$D_R(z_0) = \{z : |z_0 - z| < R\} =$$ open disc of radius R about z_0



Def^n: $\Omega \subset \mathbb{C}$ is called open if, for any $z_0 \in \Omega$, there is a disc $D_R(z_0) \subset \Omega$ (with $R > 0$).

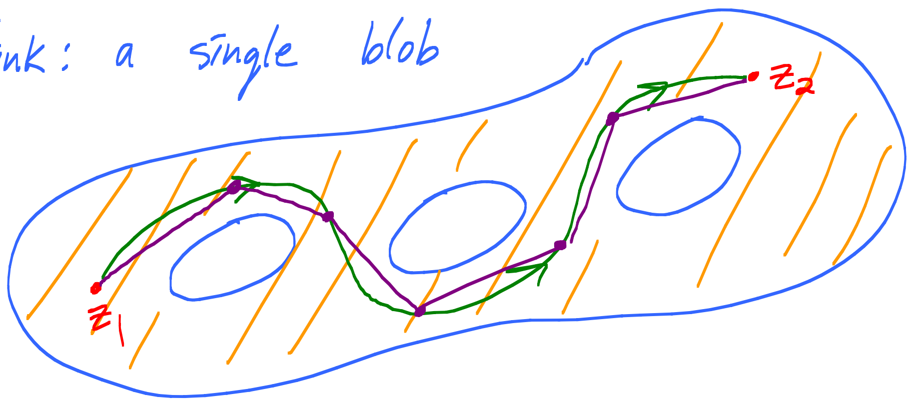
Note: R depends on z_0.

EX:  is open

ouch!  is not open (not closed either)

Important Ω's : Connected open sets $\leftarrow$ domains

Think: a single blob



Any two points in Ω can be connected in Ω by a curve that stays in Ω.

Equivalently: a polygonal path.