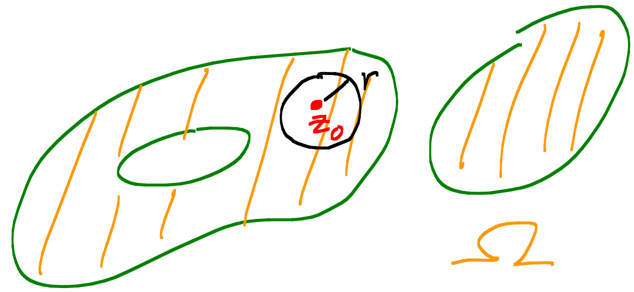
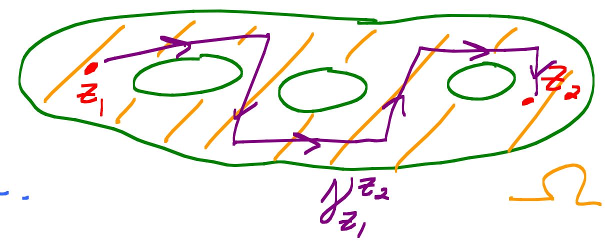
Lesson 3 Topology of $\mathbb{C}$, calculus on $\mathbb{C}$



Def'ns: 1) Ω is open if for any $z_0 \in \Omega$, there is a disc $D_r(z_0) \subset \Omega$. $(r > 0)$



2) An open set Ω is connected if any two points in Ω can be connected by a polygonal path in Ω.

3) An open connected set is called a domain.

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

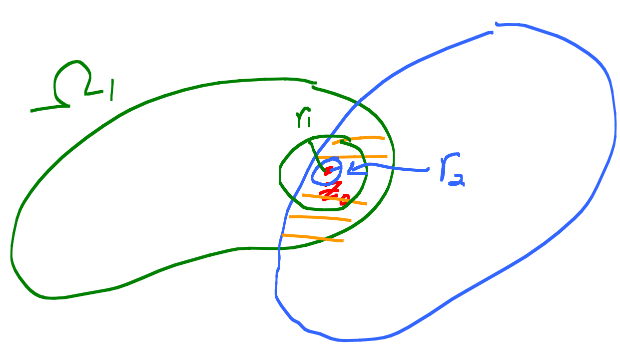
Lemma: Suppose $u(x,y)$ is continuously diff'ble fcn on a domain Ω. If $\nabla u \equiv 0$ on Ω, then u is constant on Ω.

Pf: $$u(x_2,y_2) - u(x_1,y_1) = \int_{\gamma_{z_1}^{z_2}} \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = 0$$

Fix z_1. Let z_2 slide around. So $u \equiv u(x_1,y_1)$ on Ω.

Fact: If Ω_1 and Ω_2 are open, then so are $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$.

Why: $\Omega_1 \cap \Omega_2$



Aha!

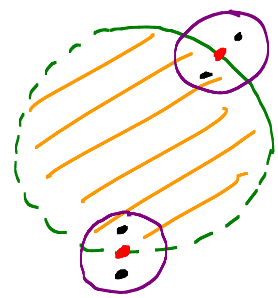
Take $r = \mathrm{Min}(r_1, r_2)$

Get $D_r(z_0) \subset \Omega_1 \cap \Omega_2$

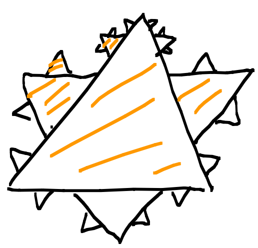
Hwk prob: Ω_1, Ω_2 domains. Is $\Omega_1 \cap \Omega_2$ a domain?

Ans. No. Draw a counterexample.

Def^n: A point z_0 is a boundary point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0$ (no matter how small), $D_\varepsilon(z_0)$ contains a point in S and a point outside S.



Sierpinski snowflake:



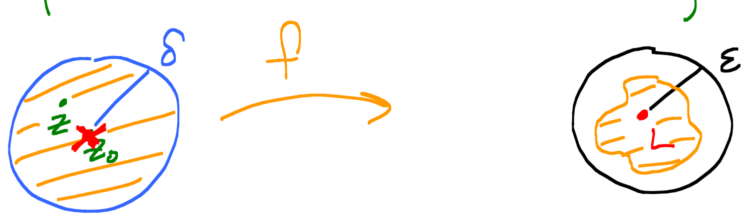
A domain with finite area and infinite perimeter. Nowhere diff'ble bndry curve!

Calculus on $\mathbb{C}$: Limits.

$$\lim_{z \to z_0} f(z) = L \quad \text{means:}$$

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \quad \text{if} \quad |z - z_0| < \delta, \quad z \neq z_0.$$



Def^{n}: f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$,

f is continuous on an open Ω if it is cont. at each pt in Ω.

Fact: All rules about limits and continuous fcns $\mathbb{R} \to \mathbb{R}$ carry over $\mathbb{C} \to \mathbb{C}$ with the same proofs!

EX: If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$,

then $\lim_{z \to z_0} f(z) g(z)$ exists and $= AB$.

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided that $B \neq 0$.

Consequence: Complex polys are continuous.

why: $$a_0 + a_1 z + a_2 \underset{\substack{\parallel \\ z \cdot z}}{z^2} + \cdots + a_N z^N$$ etc.

Rational fcns $\frac{P(z)}{Q(z)}$ are continuous away from zeroes of $Q(z)$.

Def^{n}: f is complex differentiable at z_0 if

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. If it does, call it $f'(z_0)$.

Fact: All rules of diff'tion $\mathbb{R} \to \mathbb{R}$ carry over $\mathbb{C} \to \mathbb{C}$ with same proofs!

EX: Show $(fg)' = f'g + fg'$

$$\frac{f(z)g(z) - f(a)g(a)}{z-a} = \frac{f(z)g(z) - f(z)g(a) + f(z)g(a) - f(a)g(a)}{z-a}$$

$$= f(z) \cdot \frac{g(z)-g(a)}{z-a} + g(a) \cdot \frac{f(z)-f(a)}{z-a}$$

$z \to a$:

$$f(a) \cdot g'(a) + g(a) \cdot f'(a) \quad \checkmark$$

Def^n: f is <u>analytic</u> on an open set Ω if it is complex diff'ble at every pt in Ω.

MA 425/525: Analytic fcns on domains.

Fact: Complex polys are $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire}}$. So are rational fcns away from zeroes of denom.

EX: 1) $z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i \underbrace{2xy}_{v(x,y)} \leftarrow$ complex diff'ble

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

2) $e^{z} = e^{x+iy} = e^{x}e^{iy} = \left(e^{x}\cos y\right) + i\left(e^{x}\sin y\right)$

Is e^{z} complex diff'ble? Yes!

3) $\bar{z} = \overline{(x+iy)} = x - iy$ is <u>not</u> $\mathbb{C}$-diff'ble!

$f(z) = \bar{z}$. $\quad f(re^{i\theta}) = \overline{re^{i\theta}} = re^{-i\theta}$