Lesson 4 Analytic fcns.

Lessons 1, 2, 3 due Wed. Stapled as one.

Correction: z_0 is a boundary point of S if, for every $\varepsilon > 0$, $D_\varepsilon(z_0)$ contains a pt in S and a pt outside S.

don't need $D_\varepsilon(z_0) - \{z_0\}$ here

Geometric series: $\underbrace{1 + z + z^2 + \cdots + z^N}_{S_N(z)} = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}$

Think limit $L(z)$

$$S_N(z) = 1 + z + z^2 + \cdots + z^N$$

$$- \quad z\,S_N(z) = z + z^2 + \cdots + z^N + z^{N+1}$$

$$(1-z)\,S_N(z) = 1 - z^{N+1}$$

$$S_N(z) = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}$$

Think: Error $\mathcal{E}_N(z)$

Fact: If $|z| < 1$, then

$$\lim_{N \to \infty} S_N(z) = \frac{1}{1-z}$$

Numerator estimate: $|z+w| \leq |z| + |w|$

Denominator estimate: $|z-w| \geq \big|\,|z| - |w|\,\big|$

Replace w with $-w$

Get $|z+w| \geq \big|\,|z| - |w|\,\big|$.

$$|1-z| \geq \big|\,|1| - |z|\,\big| = 1 - |z| \quad \text{if } |z| < 1$$

So $\left|\mathcal{E}_N(z)\right| = \dfrac{|z^{N+1}|}{|1-z|} \le \underbrace{\dfrac{|z^{N+1}|}{1-|z|}}$ if $|z|<1$

$$\frac{|z|^{N+1}}{1-|z|} \to 0 \text{ as } N\to\infty$$

when $|z|<1$

Note: Calculation bombs if $|z|\ge 1$.

Fact: $\lim\limits_{N\to\infty} \underbrace{S_N(z)}_{\text{Power series!}} = \dfrac{1}{1-z}$ exactly on $D_1(0)$.

Power of complex exponential:

DM: $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Famous Fourier Series Fact:

$$\underbrace{\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos N\theta}_{C_N} = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

Why: $C_N = \operatorname{Re}\left[1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{iN\theta}\right] - \frac{1}{2}$

$$= \mathrm{Re}\Big[\underbrace{1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^N}_{\frac{1-(e^{i\theta})^{N+1}}{1-e^{i\theta}}\cdot\frac{e^{-i\theta/2}}{e^{-i\theta/2}}}\Big] - \frac{1}{2}$$

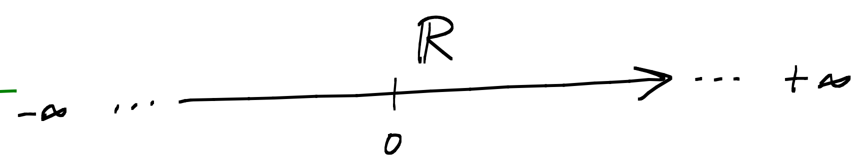
$$\underbrace{\frac{1-(e^{i\theta})^{N+1}}{1-e^{i\theta}}\cdot\frac{e^{-i\theta/2}}{e^{-i\theta/2}}}_{\frac{-\frac{1}{2i}\left(e^{-i\theta/2} - e^{i(N+\frac{1}{2})\theta}\right)}{\left(\frac{e^{-i\theta/2}-e^{i\theta/2}}{-2i}\right)}}$$

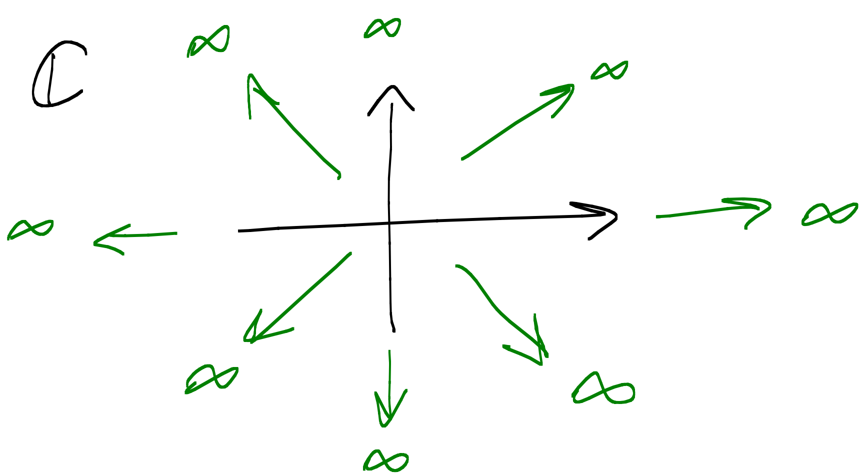
$$\left(\frac{e^{-i\theta/2}-e^{i\theta/2}}{-2i}\right) \leftarrow \sin\frac{\theta}{2} \ !$$

$$= \underbrace{\frac{\frac{1}{2}i\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos(N+\frac{1}{2})\theta - i\sin(N+\frac{1}{2})\theta\right)}{\sin\frac{\theta}{2}}}_{}$$

$$\mathrm{Re\ part} = \frac{1}{2} + \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \qquad \text{Done!}$$

The "point at infinity":





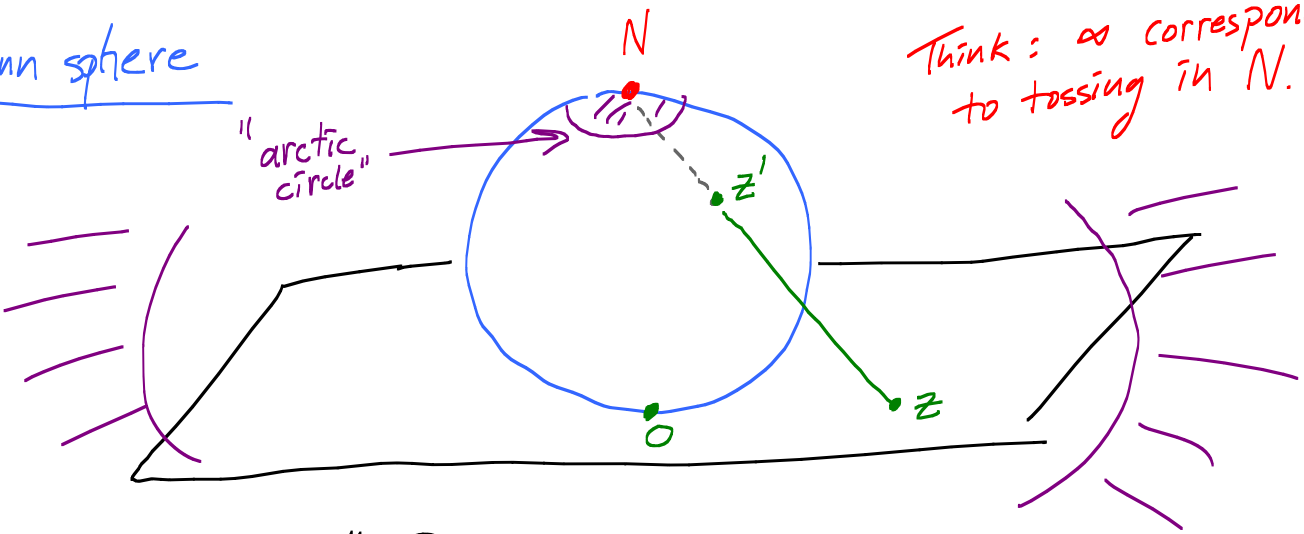
Def^n: $\lim_{z\to a} f(z) = \infty$ means $|f(z)| \to \infty$ as $z \to a$.

Same as: Given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z-a| < \delta$, $z \neq a$.

EX: $\lim_{z \to a} \frac{1}{z-a} = \infty$. So is $\lim_{z \to a} \frac{1}{(z-a)^3}$

Riemann sphere

Think: ∞ corresponds to tossing in N.



$[-\infty, \infty]$ "compactifies" $\mathbb{R}$

"Disc about ∞ in $\mathbb{C}$" $= D_\varepsilon(\infty) = \{z : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$

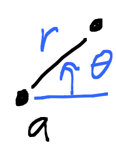
Notation: "Extended complex plane"

$$= \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

the pt at ∞

EX: $f(z) = \bar{z} = \overline{(x+iy)} = x - iy$

Not complex diff'ble at a.



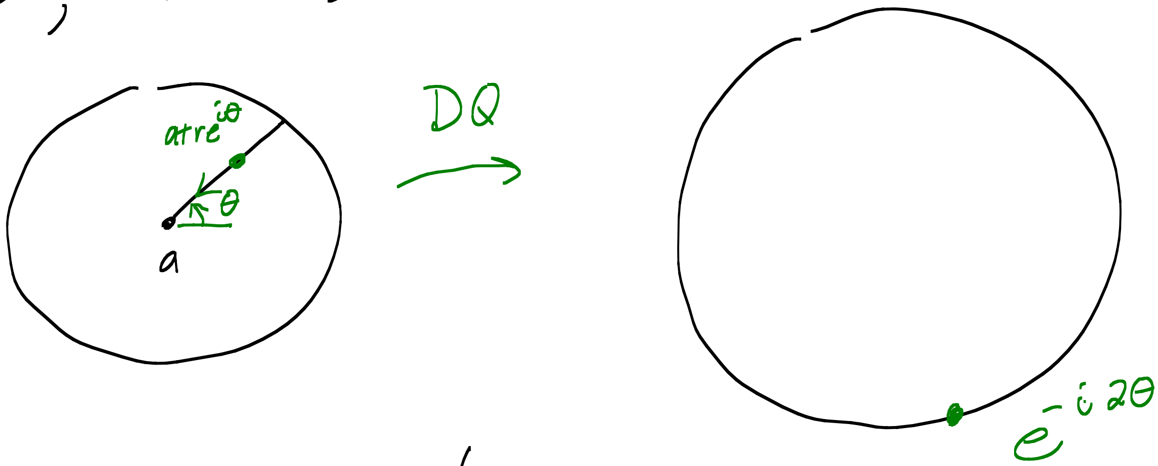
$z = a + re^{i\theta}$

think r small

$$DQ = \frac{f(z) - f(a)}{z-a} = \frac{\overline{(a+re^{i\theta})} - \bar{a}}{(a+re^{i\theta}) - a} = \frac{r \cdot \overline{e^{i\theta}}}{r\, e^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

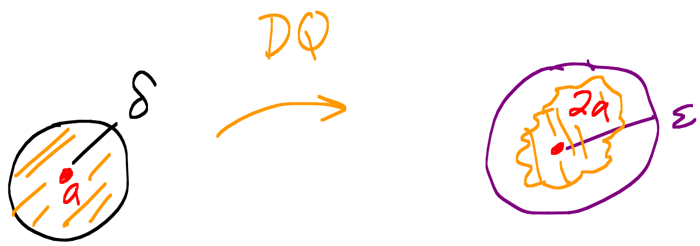
What?

As $r \searrow 0$, $a+re^{i\theta}$ slides into a.

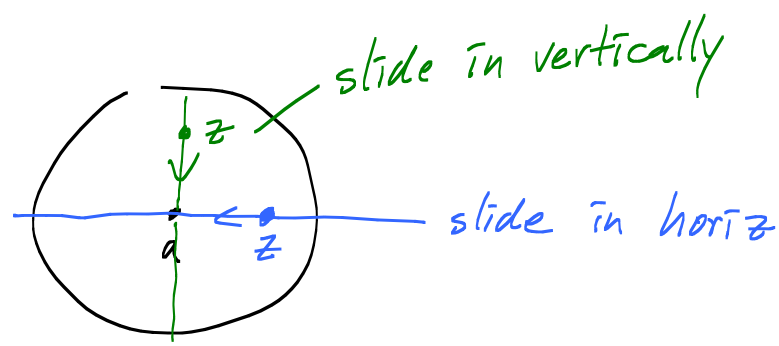


$\lim\limits_{z\to a}$ DQ does not exist!

EX: $f(z)=z^2$ $\qquad DQ = \dfrac{z^2-a^2}{z-a} = z+a \xrightarrow[\text{as } z\to a]{} 2a \leftarrow$ a single pt.



Cauchy-Riemann eqns:



Need DQ_{horiz} and DQ_{vert} need to $\to$ same thing.