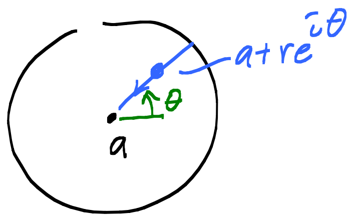
Lesson 5 The Cauchy-Riemann Equations

Read Chap 2
Lessons 4, 5, 6 due
next Wed



z^2: $DQ = \dfrac{z^2-a^2}{z-a} = z+a = (a+re^{i\theta})+a$

$= 2a + re^{i\theta} \to 2a$ as $r \to 0$

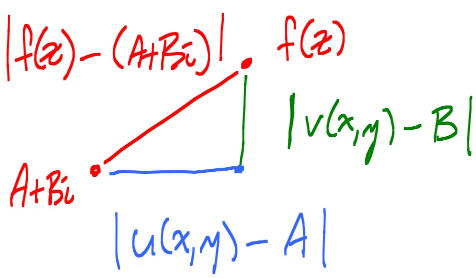
$\bar{z}$: $DQ = \dfrac{\bar{z}-\bar{a}}{z-a} = e^{-i2\theta}$

Lemma: Write $z = x+iy$, $z_0 = x_0+iy_0$

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$\lim_{z\to z_0} f(z) = A+Bi \iff \begin{array}{l} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = A \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = B \end{array}$$

Why:



Fact follows from

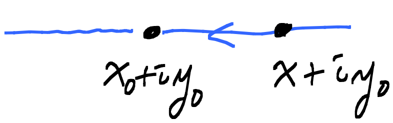
$\Longrightarrow$ (legs) < hyp

$\Longleftarrow$ Pythagorean thm

Cauchy-Riemann Eqns (C-R Eqns)

$$DQ = \frac{f(z)-f(z_0)}{z-z_0}$$

Slide in along horizontal:



$$DQ = \frac{[u(x,y_0)+iv(x,y_0)] - [u(x_0,y_0)+iv(x_0,y_0)]}{(x+iy_0)-(x_0+iy_0)}$$

$$= \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\,\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

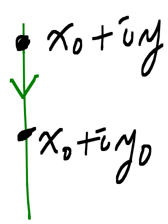
Aha! If complex DQ $\to$ limit, then the real and imag parts $\to$ limits (by Lemma).

If f is $\mathbb{C}$-diff'ble at z_0, then $\frac{\partial u}{\partial x}(x_0,y_0)$ and $\frac{\partial v}{\partial x}(x_0,y_0)$ exist and

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$$

red box #1

Vertical:



$$DQ = \frac{f(x_0+iy) - f(x_0+iy_0)}{(x_0+iy) - (x_0+iy_0)}$$

$$= \frac{u(x_0,y) - u(x_0,y_0)}{i\,(y-y_0)} + \frac{v(x_0,y) - v(x_0,y_0)}{y - y_0}$$

$$\longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)$$

$-i = \frac{1}{i}$

If f is $\mathbb{C}$-diff'ble at z_0, then $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at (x_0,y_0) and

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$$ red box #2

C-R Eqns: Equate red boxes:

Thm: If $f = u + iv$ is $\mathbb{C}$-diff'ble at z_0, then the first partials of u and v exist at (x_0,y_0) and satisfy the C-R Eqns

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{at} \quad (x_0,y_0)$$

Furthermore, $$f'(z_0) = \begin{cases} \dfrac{\partial u}{\partial x}(x_0,y_0) + i\,\dfrac{\partial v}{\partial x}(x_0,y_0) & \leftarrow \text{red box \#1} \\[2ex] \dfrac{\partial v}{\partial y}(x_0,y_0) - i\,\dfrac{\partial u}{\partial y}(x_0,y_0) & \leftarrow \text{red box \#2} \end{cases}$$

EX: $f(z) = \bar{z} = x - iy \leftarrow$ $u(x,y) = x$, $v(x,y) = -y$

Oops! $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial v}{\partial y} = -1$

C-R eqn #1 fails. (C-R eqn #2 is ok)

Can't be $\mathbb{C}$-diff'ble

EX: $$e^{x+iy} = \underbrace{(e^x \cos y)}_{u(x,y)} + i\,\underbrace{(e^x \sin y)}_{v(x,y)}$$

Hmmm. These satisfy the C-R Eqns!

We showed $\mathbb{C}$-diff'ble $\implies$ C-R Eqns.

Is the reverse true? Yes! (almost).

Consequence: e^z is $\underbrace{\text{analytic on } \mathbb{C}}_{\text{entire fcn}}$.

red box #1

and: $$\frac{d}{dz}(e^z) = \boxed{\frac{\partial}{\partial x}(e^x\cos y) + i\frac{\partial}{\partial x}(e^x \sin y)}$$

$$= e^z$$

Thm: $f(x+iy) = u(x,y) + iv(x,y)$

If u and v are C^1-smooth ← $\left\{ \begin{array}{l} u, v \text{ continuous,} \\ \text{first partials} \\ \text{exist and are} \\ \text{continuous.} \end{array} \right.$

(annoying to assume)

on a domain Ω, and

satisfy the CR Eqns on Ω, then $f = u + iv$ is analytic on Ω.

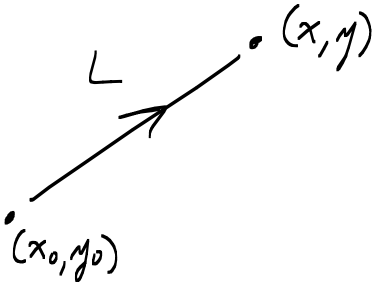
Cor: e^z is analytic

Calculus Lemma: If $u(x,y)$ is C^1-smooth, then

$$u(x,y) = \underbrace{u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)}_{\substack{\text{First order (linear)} \\ \text{approx to } u \text{ near } (x_0,y_0)}} + \underset{\substack{\uparrow \\ \text{remainder}}}{R(x,y)}$$

where $$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

Why:



$$L: \ (x_0,y_0)+t\,(x-x_0,\,y-y_0), \quad 0\le t\le 1$$

$$u(x,y)-u(x_0,y_0) = \int_L \nabla u \cdot d\vec{s}$$

$$= \int_0^1 \left(\frac{\partial u}{\partial x}\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big),\ \frac{\partial u}{\partial y}\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big) \right) \cdot \big((x-x_0),\ (y-y_0) \big)\, dt$$

$$= \int_0^1 \frac{\partial u}{\partial x}\cdot(x-x_0) + \frac{\partial u}{\partial y}(y-y_0) \quad dt$$

Hmmm: $$\int_0^1 \underbrace{\frac{\partial u}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)\,(y-y_0)}_{\text{linear part of Taylor}}\ dt$$

Aha! $$R(x,y) = \int_0^1 \frac{\partial u}{\partial x}\big(x_0+t(x-x_0),\, y_0+t(y-y_0)\big)\cdot(x-x_0) + \cdots - \frac{\partial u}{\partial x}(x_0,y_0)\,(x-x_0) + \cdots$$

and $\frac{\partial u}{\partial x}$ along L is close to $\frac{\partial u}{\partial x}(x_0, y_0)$ if (x,y) is close to (x_0, y_0).

C^1-smooth means $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ continuous. Give me an $\varepsilon > 0$.

There is a $\delta > 0$ such that

$$\Big| \underset{\substack{\uparrow \\ \text{along } L}}{\frac{\partial u}{\partial x}} - \frac{\partial u}{\partial x}(x_0, y_0) \Big| < \frac{\varepsilon}{2} \quad \text{when } \operatorname{dist}\big((x,y),(x_0,y_0)\big) < \delta$$

Same for $\frac{\partial}{\partial y}$ case.

$$|R(x,y)| = \Big| \int_0^1 \Big| \le \int_0^1 \big| \cdots \big| \, dt$$

$$\le \int_0^1 \underbrace{\Big| \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}(x_0,y_0) \Big|}_{< \frac{\varepsilon}{2}} |x - x_0| + \underbrace{\Big| \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}(x_0,y_0) \Big|}_{< \frac{\varepsilon}{2}} |y - y_0| \, dt$$

$$\le \frac{\varepsilon}{2} \big(|x - x_0| + |y - y_0| \big)$$

$$\frac{|R(x,y)|}{|z - z_0|} \le \frac{\varepsilon}{2} \Bigg(\underbrace{\frac{|x - x_0|}{|z - z_0|}}_{\le 1} + \underbrace{\frac{|y - y_0|}{|z - z_0|}}_{\le 1} \Bigg) < \varepsilon$$