Lesson 7 Harmonic conjugates

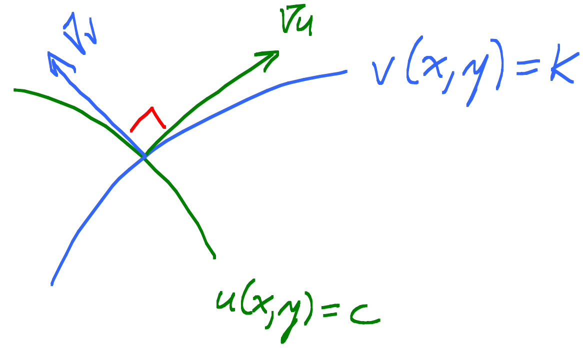
Lessons 7, 8 due next Wed
Exam 1 in class Mon., Sept. 30
covering Lessons 1-16

Great fact: Level sets of harmonic conjugates are $\perp$

Why: u harmonic. $f = u + iv$ is analytic

(v = harmonic conj for u)

C-R Eqns: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$



(Normal vector to $u(x,y)=c$ is ∇u.)

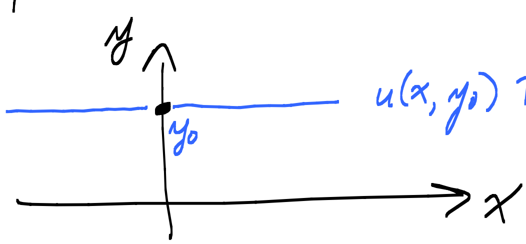
$$\nabla u \cdot \nabla v = (u_x, u_y) \cdot (\underset{-u_y}{v_x}, \underset{u_x}{v_y}) = 0$$

Fact: Steady state temperature fcn is harmonic:

Heat eqn: $\underbrace{\frac{\partial u}{\partial t}}_{\to 0 \text{ at steady state}} = c\Delta u$ So $\Delta u = 0$ at steady state

Fact: If $\frac{\partial u_1}{\partial x} \equiv \frac{\partial u_2}{\partial x}$, then $u_1 - u_2 =$ fcn of y.

Why: $u = u_1 - u_2$. $\frac{\partial u}{\partial x} = (u_1)_x - (u_2)_x \equiv 0$



$u(x, y_0)$ is const in x, but might depend on y_0.
It is a fcn of y!

[Reverse is true too. $\frac{\partial}{\partial x}[g(y)] \equiv 0$.]

EX: $u(x,y) = x^3 - 3xy^2 + y$ is harmonic, $\Delta u \equiv 0$ ✓

Get conj V.

Want $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$v_x \overset{\text{want}}{=} -u_y = -(0 - 6xy + 1) \quad (A)$$

$$v_y = u_x = 3x^2 - 3y^2 + 0 \quad (B)$$

$$\begin{cases} v_x = 6xy - 1 & (A) \\ v_y = 3x^2 - 3y^2 & (B) \end{cases}$$

Step 1: Use (A): $\quad v = \int 6xy - 1 \, dx = 6\frac{x^2}{2}y - x + \underbrace{C(y)}_{\text{arb fcn of } y}$

Why $C(y)$: If $\frac{\partial v}{\partial x} = 6xy - 1$, notice that

$$\frac{\partial}{\partial x}(3x^2y - x) = 6xy - 1 \text{ too.}$$

Preliminary fact $\Rightarrow$ $v - (3x^2y - x) \equiv \underbrace{\text{fcn of } y}_{C(y)}$

Got $\boxed{v = 3x^2y - x + \underline{C(y)}}$

Step 2: Use (B): In a different way!

(B): $\frac{\partial v}{\partial y} \overset{\text{want}}{=} 3x^2 - 3y^2$ ← Use this to pin down $C(y)$

$$\frac{\partial}{\partial y}\left(3x^2y - x + C(y)\right) = 3x^2 - 3y^2$$

$$3x^2 - 0 + C'(y) = 3x^2 - 3y^2$$ ← Miracle! all x's go away!

$$\boxed{C'(y) = -3y^2}$$

Step 3: Get $C(y)$: $C(y) = \int -3y^2\,dy = -y^3 + K$

Step 4: Write v from (A) formula

$$v(x,y) = 3x^2y - x + \left(-y^3 + K\right)$$

Fact: Harmonic conjugates are unique up to $+K$.

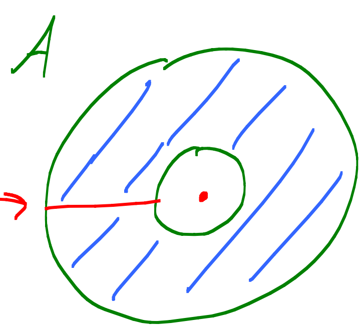
What is $f(z)$? Trick: $\begin{cases} x = \dfrac{z+\bar{z}}{2} \\ y = \dfrac{z-\bar{z}}{2i} \end{cases}$

Write out polys $u+iv$ in terms of z and $\bar{z}$.
All the $\bar{z}$'s go away!

Danger! Can find harm conj when there are no holes in the domain. Perhaps not when Ω has holes.

EX: $u = \text{Ln}\sqrt{x^2+y^2} = \text{Ln}|z|$. Find $v = \text{Arg}\, z + K$

$$= \text{Tan}^{-1}\frac{y}{x} + K$$



Ouch! → Can't hook up sides to make v harmonic on A

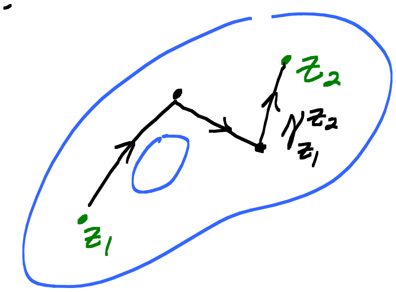
$v = \text{Arg}\, z + K$ works on A - slit

Easy Lemma: If f analytic on a domain Ω and $f' \equiv 0$ on Ω, then $f \equiv$ const on Ω.

Why: C-R Eqns red boxes: $f' = u_x + i v_x = v_y - i u_y \equiv 0$

See $\nabla u \equiv 0$ and $\nabla v \equiv 0$ if $f' \equiv 0$.

Hmmm. $u(z_2) - u(z_1) = \int_{\gamma_{z_1}^{z_2}} \nabla u \cdot d\vec{s}$

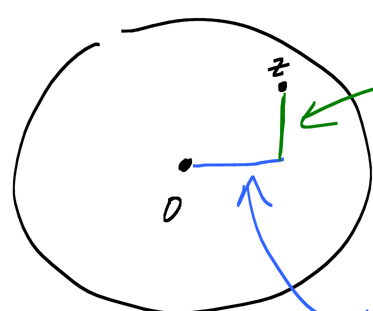


Darn! Don't know u is $\mathcal{C}^1$-smooth.

Freshman calculus fact: If $h(t)$ is continuous on $[a,b]$ and $h'(t) \equiv 0$ on (a,b), then $h(t) \equiv \text{const}$ on $[a,b]$.

Pf: MVT

On a disc:

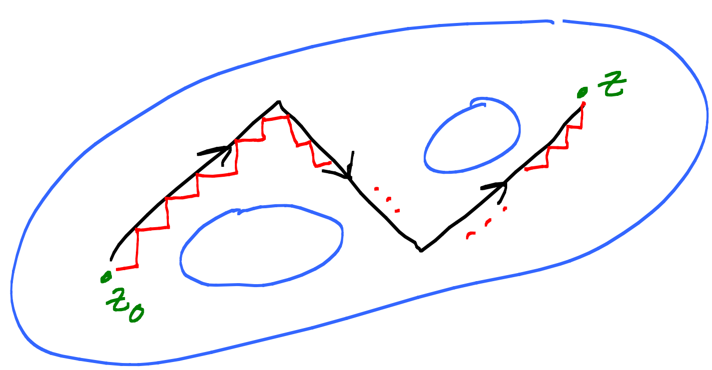


$u_y \equiv 0$ along vert, so $u = \text{const}$ along green.

$u_x \equiv 0$ along horiz, so $u = \text{const}$ on blue line.

So $u(z) = u(0)$. Same for v.

On a domain Ω:



Aha! Follow a finely cut "staircase" along poly path.

Fact: If f is analytic, then f is continuous.