Lesson 8 Analytic functions, the complex exponential

Lessons 7, 8 due Wed

Change of plan: Exam 1 in class on Monday, Sept. 30 over Lessons 1-16 (Chap 1-4)

Thm: g analytic $\Rightarrow$ g continuous.

Why: $\boxed{\dfrac{g(z)-g(z_0)}{z-z_0} - g'(z_0) = E(z)} \to 0$ as $z \to z_0$

Define $E(z_0)=0$. $E(z)$ is continuous at z_0

Solve for $g(z)$ in $\boxed{}$: $g(z) = g(z_0) + g'(z_0)(z-z_0) + E(z)(z-z_0)$

$z \to z_0$: $g(z_0) + g'(z_0)\cdot 0 + 0 \cdot 0$

So $g(z) \to g(z_0)$ as $z \to z_0$. ✓

Thm: Complex chain rule: $h(z) = f(g(z))$ $\quad w_0 = g(z_0)$

$$\boxed{f(w) = f(w_0) + f'(w_0)(w-w_0) + E(w)(w-w_0)}, \quad E(w) \to 0 \text{ as } w \to w_0.$$

$$\frac{h(z)-h(z_0)}{z-z_0} = \frac{f(\overbrace{g(z)}^{w=g(z)}) - f(\overbrace{g(z_0)}^{w_0})}{z-z_0}$$

$$= f'(w_0)\cdot\frac{(g(z)-g(z_0))}{z-z_0} + E(g(z))\frac{(g(z)-g(z_0))}{z-z_0}$$

$z \to z_0$: $f'(w_0) \quad g'(z_0) \quad + \quad \underbrace{E(w_0)}_{=0} \cdot g'(z_0)$

$$= f'(g(z_0))\, g'(z_0) \checkmark$$

Note: important to know $g(z) \to g(z_0)$ as $z \to z_0$!

Fun thing: Cook up a complex diff'ble fcn $E(z)$ such that $E'(z) = E(z)$, $E(0) = 1$.

Write $E(x+iy) = u(x,y) + i\,v(x,y)$.

$$\text{Want} \quad E'(z) = \begin{cases} u_x + i v_x & \text{(A)} \\ v_y - i u_y & \text{(B)} \end{cases} \overset{\text{want}}{=} u + iv$$

$$\text{(A)}: \qquad u_x = u \qquad v_x = v$$

$$u(x,y) = C(y)e^x \qquad v(x,y) = K(y)e^x$$

$$\text{(B)}: \qquad v_y = u \qquad -u_y = v$$

$$\underset{(1)}{K'(y) = C(y)} \qquad \underset{(2)}{-C'(y) = K(y)}$$

$$(1)': \qquad K''(y) = \underbrace{C'(y)}_{= -K(y) \text{ from } (2)}$$

$$\text{Get} \quad \boxed{K''(y) + K(y) = 0} \qquad K(y) = k_1 \cos y + k_2 \sin y$$

$$\text{Similarly:} \quad C''(y) + C(y) = 0. \qquad C(y) = c_1 \cos y + c_2 \sin y$$

Finally, initial condition $u(0,0) = 1$, $v(0,0) = 0$ pins down constants. Get e^z!

Properties of $E(z) = e^z$.

1) $E(0) = 1$
2) $E'(z) = E(z)$
3) $E(z) \neq$ for all z
4) $E(z+w) = E(z)E(w)$
5) $E(-z) = 1/E(z)$
6) $E(z) = 1 \iff z = (2\pi i)n, \ n \in \mathbb{Z}$

Lemma: If f is analytic on $\mathbb{C}$ (or $D_R(0)$) and $f'(z) = k f(z)$, then $f(z) = c\,E(kz)$ where $c = f(0)$.

Why: Assume $f' - kf = 0$ ← MA 262 ODE class!

Think: Integrating factor $u(z) = e^{-kz}$

Multiply by $E(-kz)$. [Chain rule: $\frac{d}{dz} E(-kz) = E'(-kz) \cdot [\frac{d}{dz}(-kz)]$
$= E(-kz) \cdot (-k)$]

$$\underbrace{E(-kz) f'(z) - k E(-kz) f(z)}_{\frac{d}{dz}\left[E(-kz) f(z)\right]} = 0$$

So $E(-kz) f(z) \equiv c$, a constant.

Plug in $z = 0$ to see $\boxed{c = f(0)}$.

So $\boxed{f(z) = \dfrac{f(0)}{E(-kz)}}$.

Hmmm. Is $\dfrac{1}{E(-kz)} = E(kz)$?

Sneaky way to see, yes, it is!

Start over above using $f(z) = E(kz)$. It satifies

$$f'(z) = E'(kz)\cdot k = kE(kz) = kf(z) \checkmark$$

Red box shows $E(kz) = \dfrac{1}{E(-kz)}$. Done.

Red box becomes $f(z) = f(0)E(kz)$ ✓

Very cool thing: Let $f(z) = E(z+w)$. Think $w = \text{const.}$

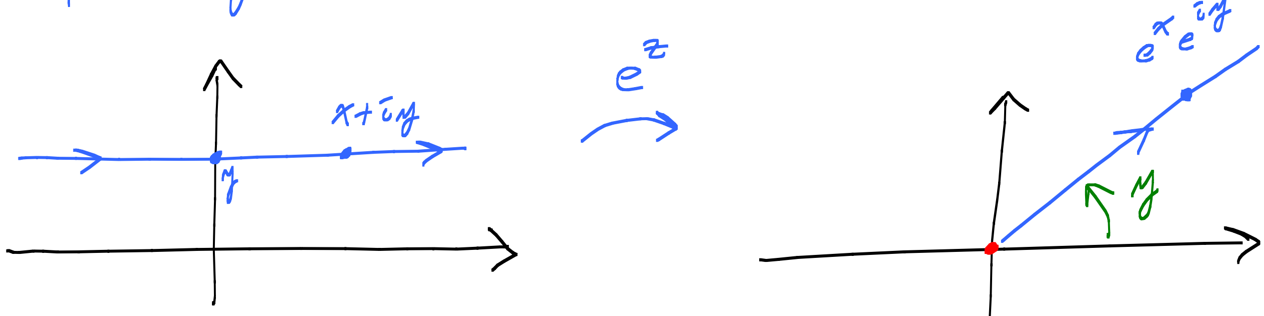
Then $$f'(z) = \underbrace{E'(z+w)}_{E(z+w)} \cdot \underbrace{\frac{d}{dz}[z+w]}_{1} = f(z)$$

Lemma $\Longrightarrow$ $f(z) = f(0)E(1\cdot z)$

$$E(z+w) = E(0+w)\cdot E(z) = E(z)E(w) \checkmark$$

Another very cool thing: All trig identities follow from these ODE facts! No triangles needed!

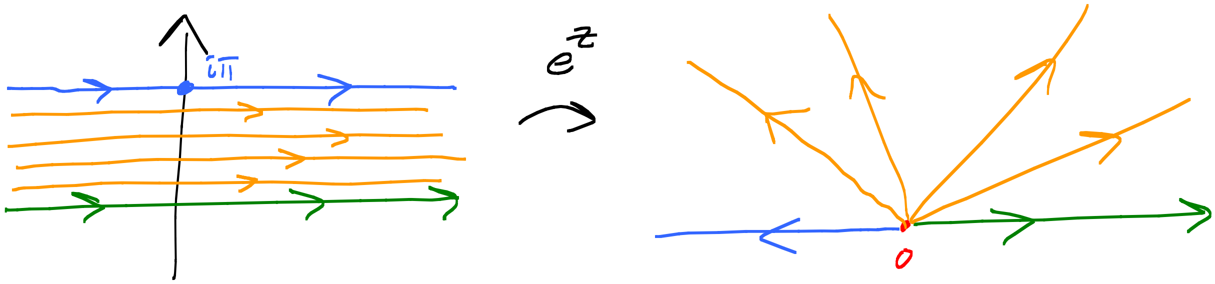
Complex log functions: Graph of e^z



$e^{x+iy} = e^x e^{iy}$

$$-\infty < x < \infty$$

$$0 < e^x < \infty$$



Aha! e^z maps strip $\{z : 0 < \operatorname{Im} z < \pi\}$ one-to-one onto the Upper half plane (UHP) $\{z : \operatorname{Im} z > 0\}$

Hmmm: