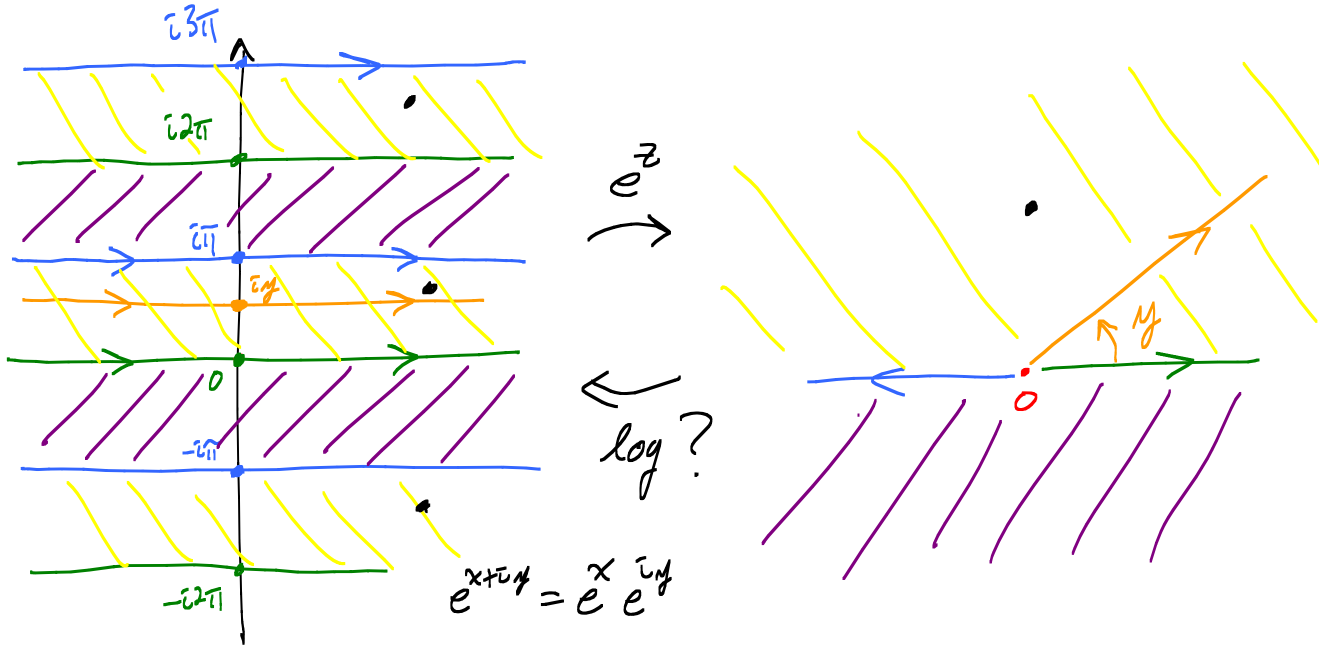
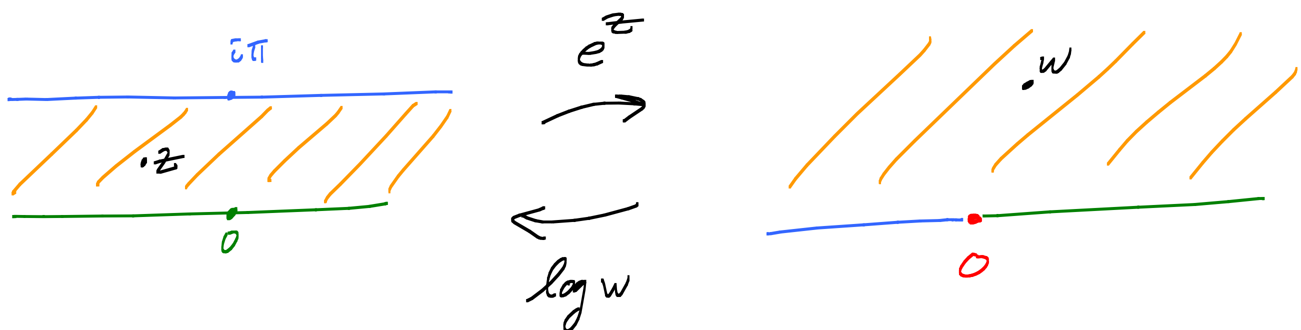
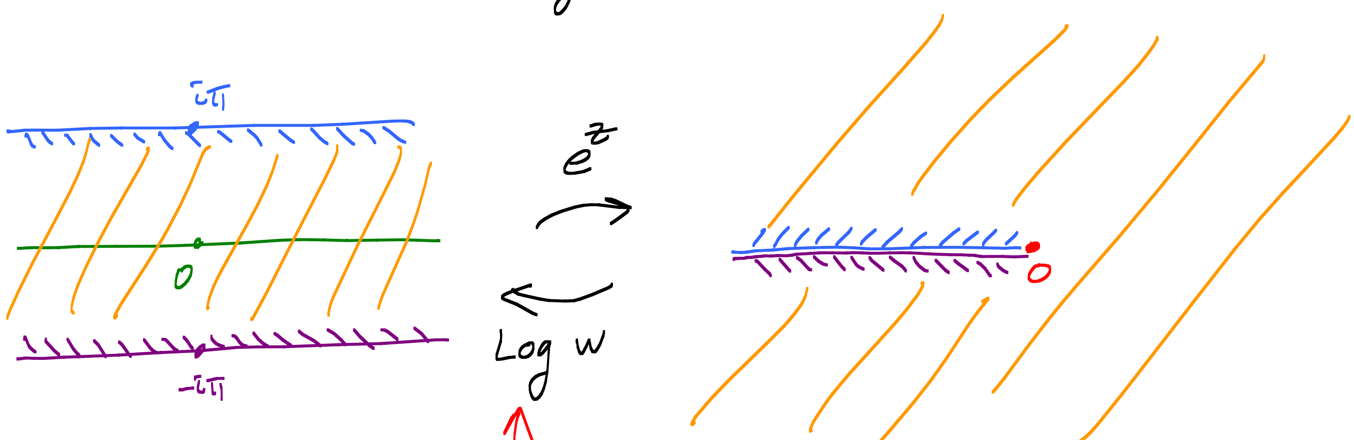
Lesson 9 Complex log functions

HWK 3: Lessons 7, 8 due Wed (9, 10, 11 next Wed)







Principal branch of complex log

Fact: e^z maps $\{z : -\pi < \operatorname{Im} z < \pi\}$ one-to-one onto

$\mathbb{C} - (-\infty, 0]$ ← negative real axis plus 0

Think: $w = e^z$. Then $z = \log w$ ← ∞ many choices!

$$w = e^{x+iy} = e^x e^{iy}$$ ← Aha! $y \in \arg w$

Aha! $e^x = |w|$ ← $x = \operatorname{Ln}|w|$

My def^n: $\log w = \{ z : e^z = w \}$

$$= \{ \operatorname{Ln}|w| + i\theta \ : \ \theta \in \arg w \}$$

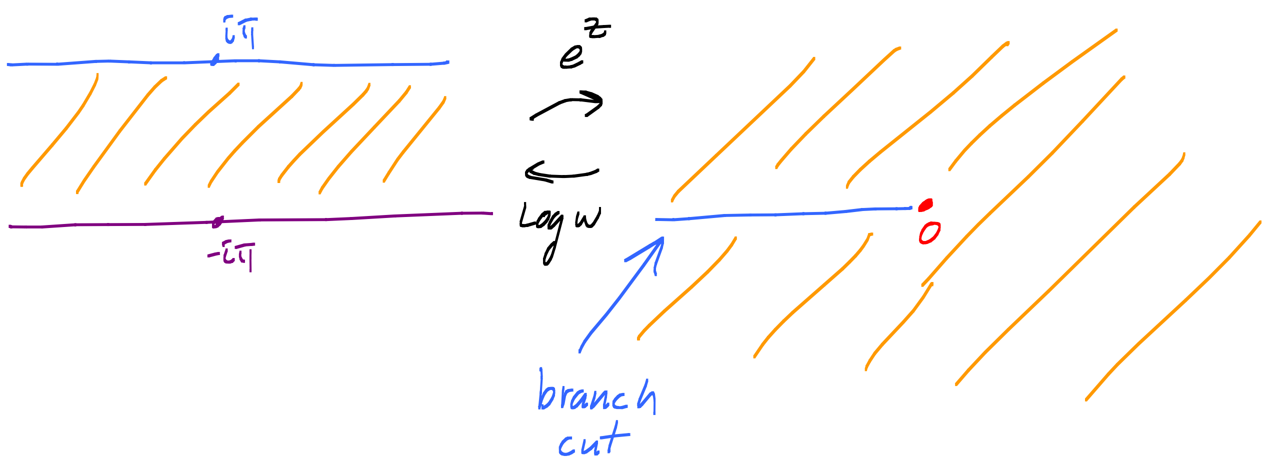
$$= \{ \operatorname{Ln}|w| + i\theta : \theta = \operatorname{Arg} w + 2\pi n, n \in \mathbb{Z} \}$$

$\operatorname{Log} w$ = Principal branch $= \operatorname{Ln}|w| + i \operatorname{Arg} w$

Principal arg

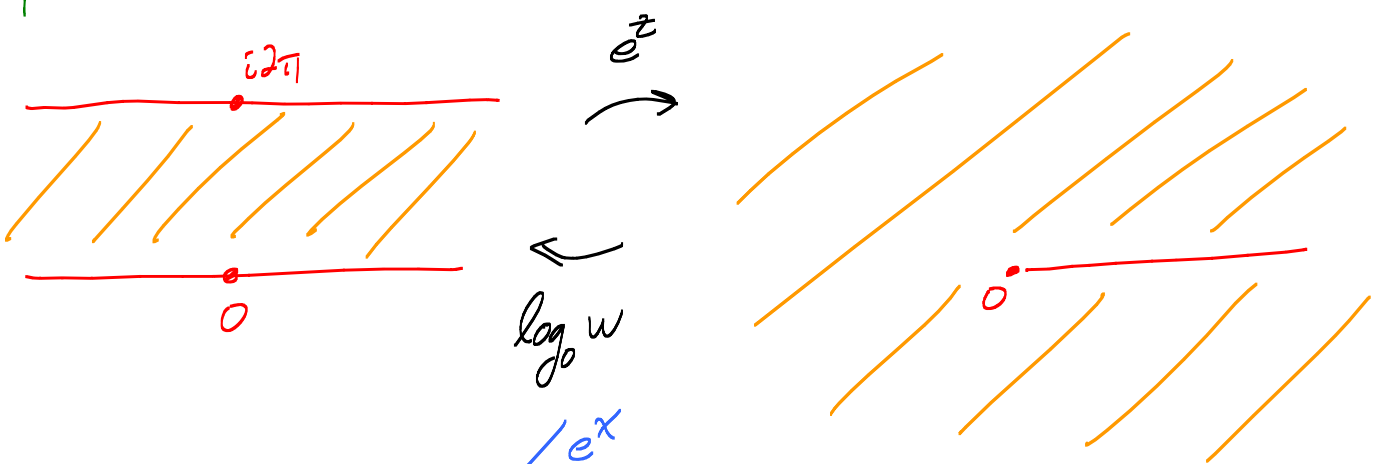
where $-\pi < \operatorname{Arg} w \le \pi$

π side wins along cut!

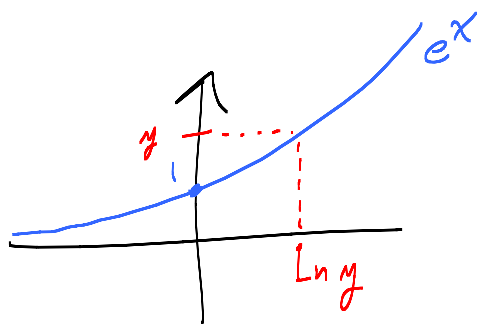


My def^n: $\log_\alpha w = \mathrm{Ln}|w| + i\theta$ where $\theta \in \arg w$ with $\alpha < \theta < \alpha + 2\pi$

Popular branch:



Real



Fact: $\mathrm{Log}\, w$, $\log_\alpha w$ are analytic on orange stuff. (domain)

Why: Polar C-R Eqns $\quad f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$

$$\log_\alpha re^{i\theta} = \mathrm{Ln}\, r + i\theta$$

$$\begin{cases} U_r = \frac{1}{r} V_\theta \\ V_r = -\frac{1}{r} U_\theta \end{cases}$$

What is $\frac{d}{dz}(\log_\alpha z)$?

Always true: $e^{\log z} = z$

Not always that $\log e^z = z$!

$$\log e^z = z + i2\pi n \text{ for some } n \in \mathbb{Z}.$$

$$\frac{d}{dz}(\text{Always true}) = \frac{d}{dz}\Big(\underbrace{e^{\log z}}_{z}\Big) = 1$$

$$= \frac{d}{dz} E(\log z)$$

$$= \underbrace{E'(\log z)}_{E(\log z) = z} \cdot \frac{d}{dz} \log z$$

See that $\boxed{\dfrac{d}{dz}(\log_\alpha z) = \dfrac{1}{z}}$

__Warning:__ Must be very careful trying things like

$$\mathrm{Log}\, z_1 z_2 = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 + i2\pi n \quad n \in \mathbb{Z}$$

but it __is__ true if $z_1, z_2 \in$ first quadrant

__Fantastic fact:__ Zoo of real fcns $\sin$, $\cos$, e^x, $\ln x$, $\sin^{-1}$, $\sinh$, $\sinh^{-1}$

collapse down to __two__ $\mathbb{C} \to \mathbb{C}$!

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \to \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Def^n: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\tan z = \dfrac{\sin z}{\cos z}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ etc.

Play around: $\cos^{-1} w = z$?

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\left(w = \frac{e^{iz} + \frac{1}{e^{iz}}}{2} \right)$$ mult by $2e^{iz}$

Get: $(e^{iz})^2 - 2w\,e^{iz} + 1 = 0$

Quadratic in e^{iz}!

So $$e^{iz} = \frac{-(-2w) + \sqrt{4w^2 - 4}}{2}$$

$$= w + \sqrt{w^2 - 1}$$

and $$iz = \log\left(w + \sqrt{w^2 - 1}\right)$$

$$\boxed{\cos^{-1} w = -i \log\left(w + \sqrt{w^2 - 1}\right)}$$

Def^n: $\quad \sinh z = \dfrac{e^z - e^{-z}}{2}$ etc.

$\sinh^{-1} w$ = simple combination of log and algebraic fcns like $\sqrt{w^2 \pm 1}$

Fantastic fact: All the "elementary functions" are given by combinations of e^z, $\log z$, and algebraic fcns.

Liouville's Thm: e^{z^2} does not have an elementary fcn for an antiderivation.