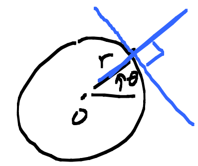
# Lesson 10: Inverse functions, complex powers and roots

9, 10, 11 due Wed

## Polar C-R Eqns:

$$\begin{cases} f(x+iy) = u(x,y) + i\,v(x,y) \\ f(re^{i\theta}) = U(r,\theta) + i\,V(r,\theta) \end{cases}$$



$$U(r,\theta) = u(r\cos\theta,\ r\sin\theta)$$
$$V(r,\theta) = v(r\cos\theta,\ r\sin\theta)$$

Chain rule:

$$U_r = u_x \cos\theta + u_y \sin\theta$$
$$U_\theta = u_x \cdot (-r\sin\theta) + u_y \cdot (r\cos\theta)$$
$$\tfrac{1}{r}U_\theta = -u_x \sin\theta + u_y \cos\theta$$

Similarly for $V_r$, $V_\theta$.

$$(*) \quad \begin{bmatrix} U_r & V_r \\ \frac{1}{r}U_\theta & \frac{1}{r}V_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \overset{!}{\Longleftarrow} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \text{ C-R Eqns}$$

Polar C-R Eqns!

**Fact:**
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

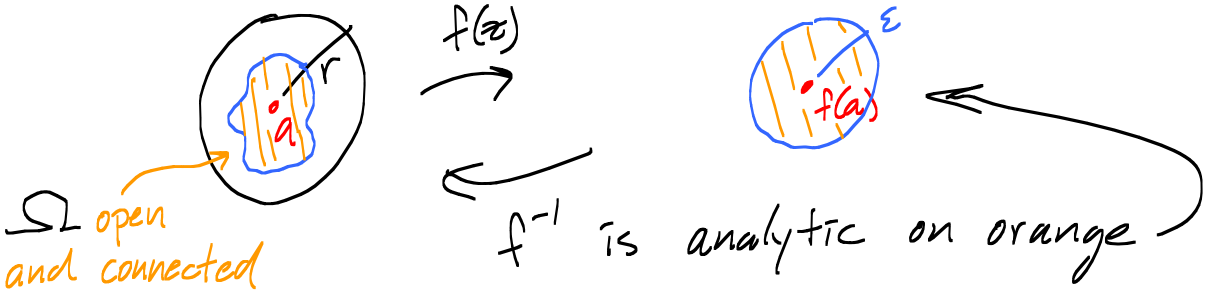
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Aha! Multiply (*) on left by

Same reasoning shows polar C-R $\Longrightarrow$ rectangular C-R.

# Inverses of analytic fcns are analytic:

**Complex inverse fcn thm:** If $f$ is analytic on $D_r(a)$ and $f'(a) \neq 0$, then $f^{-1}$ is analytic near $f(a)$.




and both are 1-1 onto.

Why: Follows from $\mathbb{R}^2 \to \mathbb{R}^2$ inverse fcn thm:

$$(x,y) \longmapsto (u(x,y), v(x,y))$$

C-R Eqns

Jacobian matrix:
$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

$$= \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$$

$$\det J = (u_x)^2 + (v_x)^2 = |\underbrace{u_x + i v_x}_{f' \text{ (red box \#1)}}|^2 = |f'|^2$$

Aha! det $\neq 0$ means inverse is diff'ble and

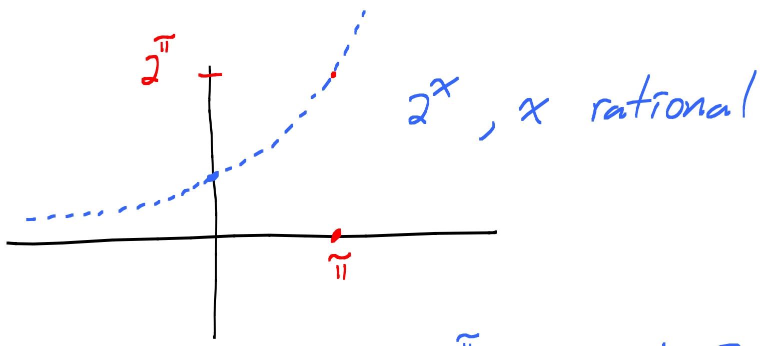
$$\text{Jacobian of inverse} = J^{-1} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{-1}$$

$$= \frac{1}{|f'|^2} \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$ See C-R Eqns!

Conclude inverse is analytic.

**Prob: What is $i^i$?** Hmmm. $e^i = \cos 1 + i \sin 1$

$2^\pi = ?$ $\qquad 2^n = \underbrace{2 \cdot \cdots 2}_{n\text{-times}}$ $\qquad 2^{p/q} = (2^p)^{1/q}$ ← q-th root




$$2^\pi = \lim_{\frac{p}{q} \to \pi} 2^{p/q} = e^{\ln 2^\pi} = e^{\pi \ln 2}$$

**Complex version**: "$z^w$" $= e^{w \log z}$ ← $\infty$ many choices for $\log z$!

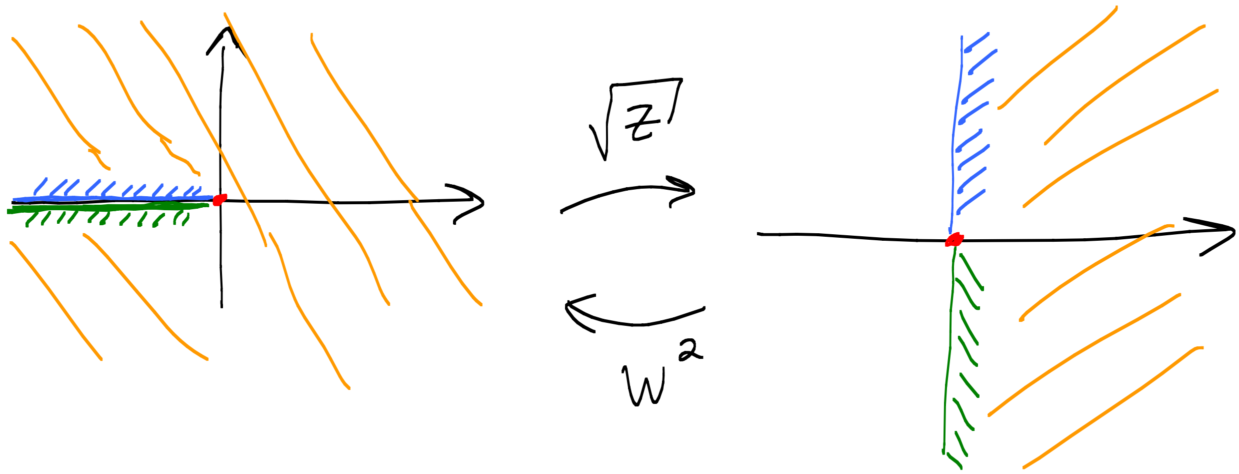
Principal branch of $z^w = e^{w \,\mathrm{Log}\, z}$ ← Principal branch of log

$$\sqrt{z} = e^{\frac{1}{2}\mathrm{Log}\, z} = e^{\frac{1}{2}(\ln|z| + i\theta)}$$

$\theta \in \arg z$, $-\pi < \theta \leq \pi$

$$= e^{\frac{1}{2}\ln|z|} \, e^{i\frac{1}{2}\theta}$$

$$= |z|^{1/2} \, e^{i\theta/2}$$



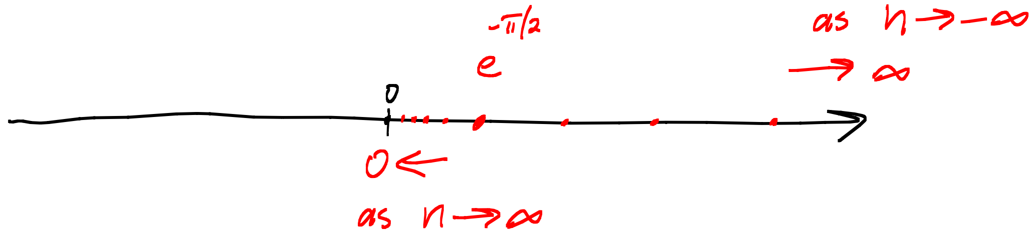

EX: $"i^i" = e^{i \log i} = e^{i(\underbrace{\mathrm{Ln}|i|}_{0} + i(\frac{\pi}{2} + 2\pi n))}$

Principal Arg $i$

$= e^{-(\frac{\pi}{2} + 2\pi n)} = e^{-\frac{\pi}{2}} \, e^{-2\pi n}$

Principal value: $i^i = e^{-\pi/2}$

What the heck?! $\infty$ many values of "$i^i$".




all <u>real</u> and $> 0$, some $\to 0$, some $\to \infty$!

<u>Easy facts</u>: $z^n$ only has one value

$z^n = e^{n \log z} = e^{n(\mathrm{Ln}|z| + i(\theta + 2\pi m))}$

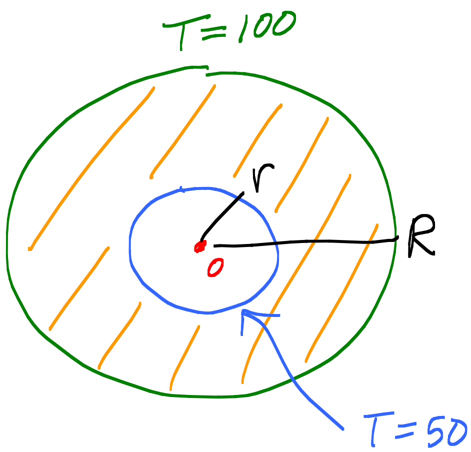
$= \underbrace{e^{n \mathrm{Ln}|z|} \, e^{in\theta}}_{z^n} \, e^{i 2\pi n m}$ ✓

$e^{i2\pi nm} = 1$

$z^{p/q} = e^{\frac{p}{q}\log z}$ has $q$ possible values.

## 3.4 Washers, wedges, walls.

$\text{Log}\, z = \underbrace{\text{Ln}|z|}_{\text{harmonic!}} + i\, \underbrace{\text{Arg}\, z}$ is analytic



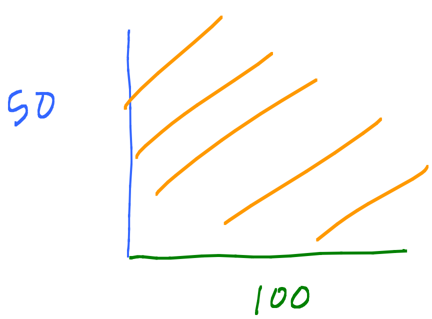

$u = \text{Ln}|z|$ ← Good idea!

$u = A\,\text{Ln}|z| + B$ ← Better idea!

Aha! Need

$$\begin{cases} A\,\text{Ln}R + B = 100 \\ A\,\text{Ln}\,r + B = 50 \end{cases}$$

Solve. Get $A, B$. $u = A\,\text{Ln}|z| + B$ solves heat prob.




$u = \text{Arg}\, z$ ← Good idea

$u = A\,\text{Arg}\, z + B$ ← Better idea!

Need: On green: $A \cdot 0 + B = 100$

On blue: $A \cdot \frac{\pi}{2} + B = 50$

Solve for $A, B$. Get sol$^n$ $A\,\text{Arg}\, z + B$