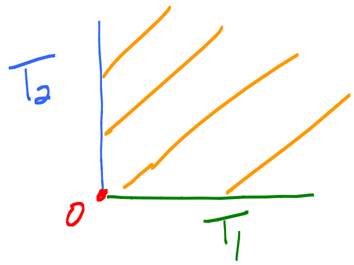
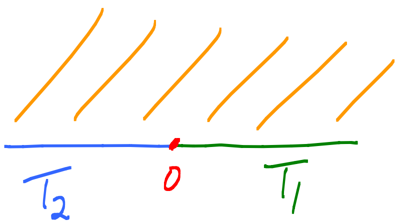
Lesson 11 Complex integration, Chap 4



$u = A + B\,\text{Arg}\, z$

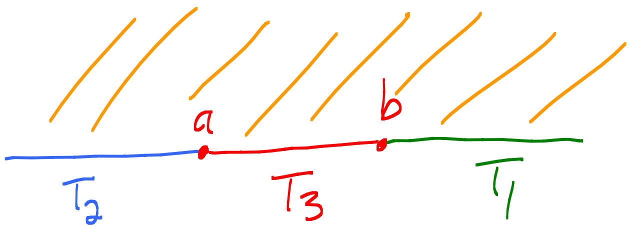
On T_1 part: Need $A + B\cdot 0 = T_1$

On T_2 : Need $A + B\cdot\frac{\pi}{2} = T_2$



On T_1 : $A + B\cdot 0 = T_1$

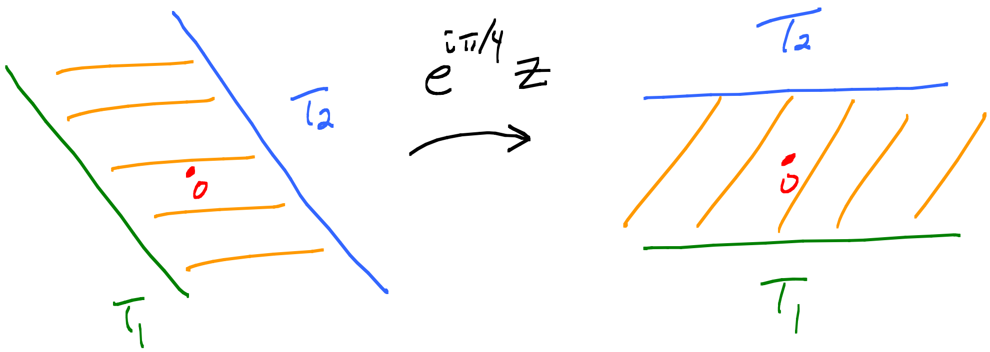
On T_2 : $A + B\cdot\pi = T_2$



Try $u = A + B\,\text{Arg}(z-a) + C\,\text{Arg}(z-b)$

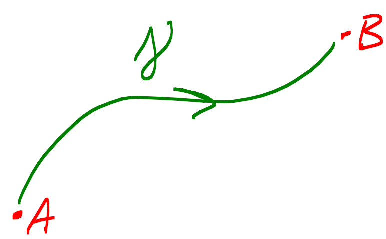
Get 3 equs for bound conds.

HWK hint:



$u = A + B\,\text{Im}\, z$

Curves in $\mathbb{C} \approx \mathbb{R}^2$

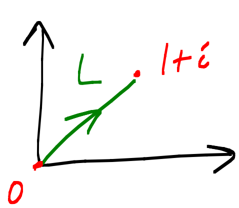


γ : $(x(t), y(t))$,

$a \le t \le b$

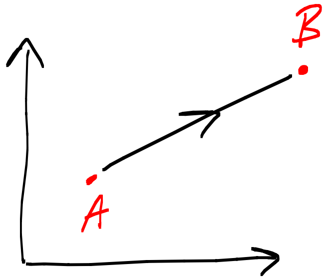
Write $z(t) = x(t) + i\,y(t)$ $\qquad z : [a,b] \longrightarrow \mathbb{C}$

EX:



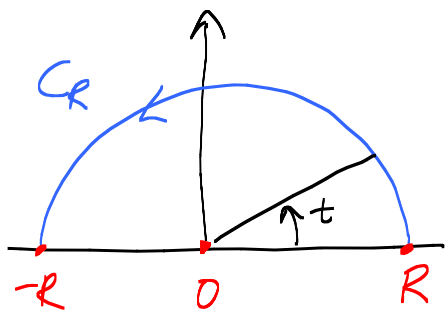
L : $z(t) = t(1+i)$, $0 \le t \le 1$

EX:



$$z(t) = A + t(B-A)$$
$$0 \le t \le 1$$
$$(A, B \text{ complex \#s})$$

EX:



$$C_R : z(t) = Re^{it}$$
$$0 \le t \le \pi$$

Two kinds of fcns: 1) $f : \Omega \to \mathbb{C}$ analytic

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

2) $z(t)$ $\quad z : [a,b] \to \mathbb{C}$ $\quad z(t) = x(t) + i y(t)$

Define $z'(t_0) = \lim_{t \to t_0} \dfrac{z(t) - z(t_0)}{t - t_0}$

$$= \lim_{t \to t_0} \left[\frac{x(t) - x(t_0)}{t - t_0} + i \frac{y(t) - y(t_0)}{t - t_0} \right]$$

$$\boxed{z'(t_0) = x'(t_0) + i y'(t_0)}$$

Two kinds of integrals:

1) Easiest kind:

$$\int_a^b z(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n)\Delta t$$

$$= \lim_{\Delta t \to 0} \left[\sum_1^N x(t_n)\Delta t + i \sum_1^N y(t_n)\Delta t \right]$$

$$\boxed{\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt}$$

assuming $x(t), y(t)$ are continuous or piecewise continuous.

Basic inequality #1 :

$$\left| \int_a^b z(t)\,dt \right| \leq \int_a^b |z(t)|\,dt \leq \left(\max_{a \leq t \leq b} |z(t)| \right)(b-a)$$

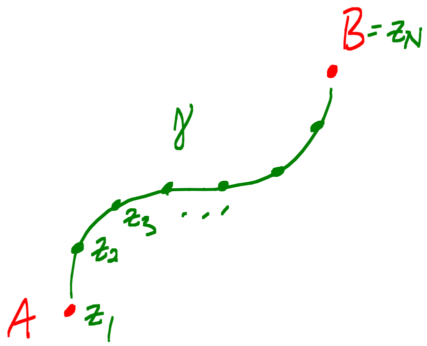
$$\|$$

$$\left| \lim \sum z(t_n)\Delta t \right| \leq \lim \left(\sum_1^N |z(t_n)|\,\Delta t \right) \checkmark$$

Baby lemma : $\displaystyle\int_a^b k\,z(t)\,dt = k\int_a^b z(t)\,dt$, $k \in \mathbb{C}$.

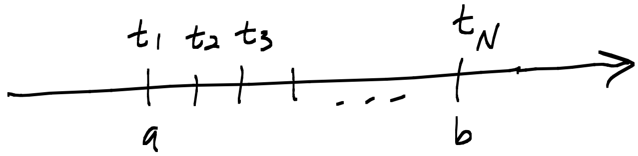
Why: Write $k = A + Bi$, $z(t) = x(t) + iy(t)$. Write out both sides. $\checkmark$

Integration of $f(z)$ along γ in $\mathbb{C}$:



$$\int_\gamma f\,dz = \lim \sum_{1}^{N-1} f(z_n)\,\underbrace{\Delta z_n}_{z_{n+1}-z_n}$$

$\gamma: z(t),\ a \le t \le b$



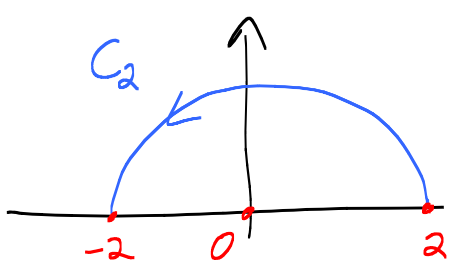
$$\int_\gamma f\,dz = \lim \sum f(z(t_n))\left(\frac{z(t_{n+1})-z(t_n)}{\Delta t}\right)\Delta t$$

$$\longrightarrow \int_a^b f(z(t))\ z'(t)\ dt$$

<u>Def^n</u>:
$$\int_\gamma f\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\ dt}_{\text{Think } = dz}$$

<u>why</u>: $\dfrac{dz}{dt} = z'(t)$

<u>EX</u>:



$C_2: z(t) = 2e^{it},\ 0 \le t \le \pi$

$= 2\cos t + i\,2\sin t$

$z'(t) = -2\sin t + i\,2\cos t$

$\underline{z'(t) = 2i\,e^{it}}$ ✓

$$\int_{C_2} e^{z}\,dz = \int_0^{\pi} e^{2e^{it}}\left(2i\,e^{it}\right)dt$$

$$= \int_0^{\pi} e^{2\cos t + i2\sin t} \cdot 2ie^{it}\, dt$$

$$= 2i \int_0^{\pi} e^{2\cos t} \underbrace{e^{i(2\sin t + t)}}_{\cos(2\sin t + t) + i \sin(2\sin t + t)} dt$$

$$= \text{Real} \int + i \text{ Real} \int \quad \text{Ugh!}$$

Cool thing: $\displaystyle\int_{C_2} e^z\, dz = \int_{C_2} \frac{d}{dz}\left[e^z\right] dz = e^z \Big|_{\text{START}}^{\text{END}}$

$$= e^{-2} - e^{2}$$

Fund Thm of Calc!

<u>Two Chain rules:</u> 1) f, g analytic. $h(z) = f(g(z))$

$$h'(z) = f'(g(z))\, g'(z)$$

2) f analytic, $z(t) : [a,b] \to \mathbb{C}$

$$\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$$

<u>Why 2:</u> $f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$

$$\frac{d}{dt} f(z(t)) = \Big[u_x \dot{x} + \underset{-v_x}{\overset{}{u_y}}\, \dot{y}\Big] + i\Big[v_x \dot{x} + \underset{u_x}{v_y}\, \dot{y}\Big]$$

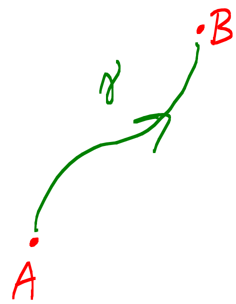
C-R Eqns

$$= \underbrace{\left[u_x + i v_x\right]}_{\substack{\text{red box \#1!} \\ = f'}} \cdot \underbrace{\left[\dot{x} + i \dot{y}\right]}_{z'(t)}$$

Two Fundamental Theorems of Calculus:

1) $$\int_a^b z'(t)\,dt = z(b) - z(a)$$ Easy!

Write $\int_a^b z'(t)\,dt = \int_a^b x'(t)\,dt + i \int_a^b y'(t)\,dt$

2) 

$$\int_\gamma f'\,dz = f(B) - f(A)$$

Two Basic Estimates:

1) $$\left| \int_a^b z(t)\,dt \right| \le M\,(b-a)$$

$$M = \max_{a \le t \le b} |z(t)|$$

2) $$\left| \int_\gamma f\,dz \right| \le \left(\max_\gamma |f| \right) \cdot \text{Length}(\gamma)$$