12, 13, 14 due next Wed
9, 10, 11 due Wed

Lesson 12 Complex integrals

Two kinds of fcns $\begin{cases} w = f(z) & f : \Omega \to \mathbb{C} \\ z(t) = x(t) + i y(t) & z : [a,b] \to \mathbb{C} \end{cases}$

Two kinds of derivatives $\begin{cases} f'(z) & \text{(complex deriv)} \\ z'(t) = x'(t) + i y'(t) \end{cases}$

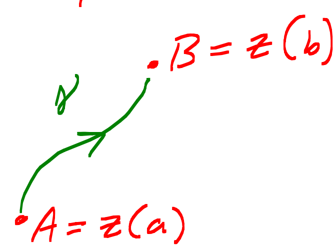
Two kinds of integrals $\begin{cases} \int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i \int_a^b y(t)\,dt \\ \int_\gamma f\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz} \end{cases}$

Two kinds of Chain rules: $\begin{cases} h(z) = f(g(z)) : \quad h'(z) = f'(g(z))\,g'(z) \\ \dfrac{d}{dt} f(z(t)) = f'(z(t))\,z'(t) \end{cases}$

Two fundamental theorems of calculus:

1) $\displaystyle\int_a^b z'(t)\,dt = z(b) - z(a)$ (Easy)

2) $\displaystyle\int_\gamma f'\,dz = f(\text{END}) - f(\text{START})$



Why (2): $\displaystyle\overset{\shortparallel}{\int_a^b} \underbrace{f'(z(t))\,z'(t)}_{\frac{d}{dt}\left[f(z(t))\right]}\,dt \underset{\substack{\uparrow \\ \text{F. Thm} \\ \text{Calc} \\ \#1}}{=} f(z(b)) - f(z(a))$ ✓

Two Basic Estimates: 1) $\displaystyle\left| \int_a^b z(t)\,dt \right| \le \int_a^b |z(t)|\,dt$

$\le M \cdot (b-a)$

where $M = \underset{a \le t \le b}{\text{Max}} |z(t)|$

(why: Riemann sums, Δ Ineq)

2) $$\left| \int_\gamma f \, dz \right| \le \left(\underset{tr(\gamma)}{\text{Max}} |f| \right) \cdot \text{Length}(\gamma)$$

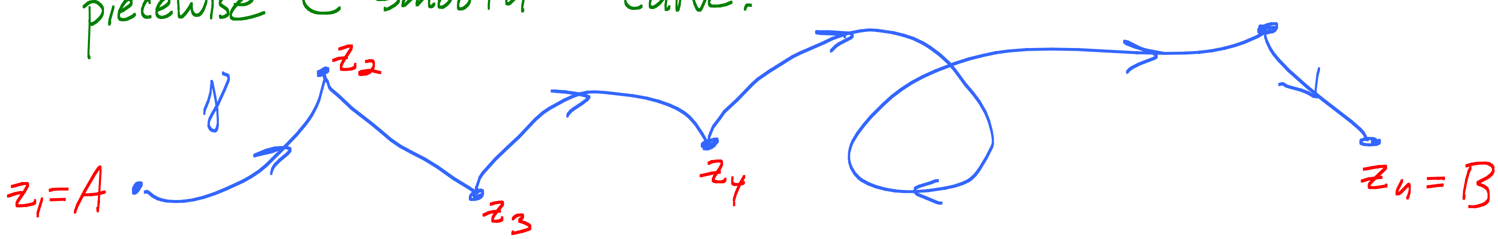
where $tr(\gamma) =$ "Trace of γ" $= \{ z(t) : a \le t \le b \}$.

<u>why:</u> $$\left| \int_\gamma f \, dz \right| = \left| \int_a^b f(z(t)) \, z'(t) \, dt \right|$$

$$\underset{\substack{\uparrow \\ \text{Basic Est} \\ \#1}}{\le} \int_a^b \underbrace{|f(z(t))|}_{\substack{\le M \\ \| \\ \underset{a \le t \le b}{\text{Max}} |f(z(t))|}} \; \underbrace{|z'(t)|}_{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} \, dt$$

$$\le M \cdot \underbrace{\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} \, dt}_{\text{Length}(\gamma)}$$

<u>Important fact:</u> I've been assuming f and γ are $\mathcal{C}^1$-smooth, but all our facts hold if γ is a "piecewise $\mathcal{C}^1$-smooth" curve:



Why: $\int_\gamma f' \, dz = \sum \int_{\gamma_k} f' \, dz$

$$= \left[f(z_2) - f(z_1) \right] + \left[f(z_3) - f(z_2) \right] + \cdots + \left[f(z_n) - f(z_{n-1}) \right]$$

Aha! Get pairwise cancellation, except for first and last ones. $= f(z_n) - f(z_1)$ ✓

Coming soon: Cauchy Theorem: f analytic on a domain Ω with no holes. Then

$$\int_\gamma f \, dz = 0 \quad \text{for all closed curves } \gamma \text{ in } \Omega.$$

Funny thing: The reverse is true too. Must study $\int_\gamma f \, dz$!

Read $\underbrace{4.4\text{b}}_{\text{Green's Thm}}$ and skip $\underbrace{4.4\text{a}}_{\text{Homotopy of curves}}$ for now.

Big fact: Suppose f is analytic on a domain Ω.

$f(z)$ has an analytic antiderivative $\iff \int_\gamma f \, dz = 0$ for any closed curve γ in Ω

Why: ($\Longrightarrow$) easy direction. If $f(z) = F'(z)$, then

$$\int_\gamma f\,dz = \int_\gamma F'\,dz \underset{\text{Fund. Thm. Calc.}}{=} F(\text{END}) - F(\text{START}) = 0$$

Fund. Thm. Calc.

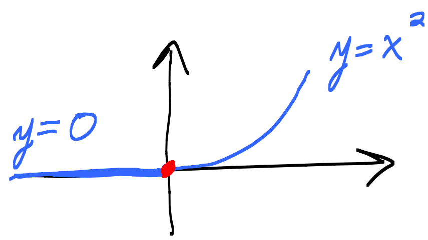
same points, γ closed

Sticky point: Need F' to be continuous to use F. Thm. Calc.

Thing that saves the day: Advanced fact: If f is complex diff'ble on $D_r(a)$, then so is f'! So is f'', f''', ... Wow! [Note: f' exists $\Rightarrow$ f cont., f'' exists $\Rightarrow$ f' cont., etc.]

f analytic $\Rightarrow$ f is infinitely complex diff'ble and all derivatives are cont.

Way false $\mathbb{R} \to \mathbb{R}$:



$\mathcal{C}^1$-smooth but not twice diff'ble at $x=0$

We get around this like so:

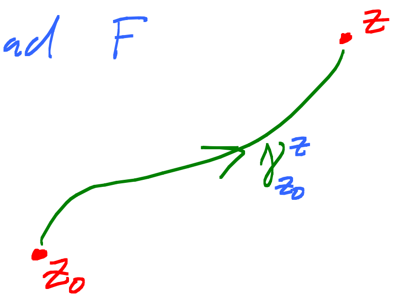
1) f analytic means f complex diff'ble on domain Ω

2) $f = u + iv$ analytic means u, v $\mathcal{C}^1$-smooth and satisfy C-R Eqns. [Then f' exists and f' is continuous.]

Advanced fact: (1) and (2) are equiv. We'll take (2) as our def^n.

Back to big fact: ($\Leftarrow$). Assume $\int_\gamma f\,dz = 0$ for all closed γ in Ω. Want F with $F' = f$.

Hmmm. If I had F



$$\int_{\gamma_{z_0}^{z}} f\,d\zeta = \int_{\gamma_{z_0}^{z}} F'\,d\zeta$$

$$= F(z) - F(z_0)$$

Aha! If I had F, see that

$$F(z) = \int_{\gamma_{z_0}^{z}} f\,dw \quad \text{would work!}$$

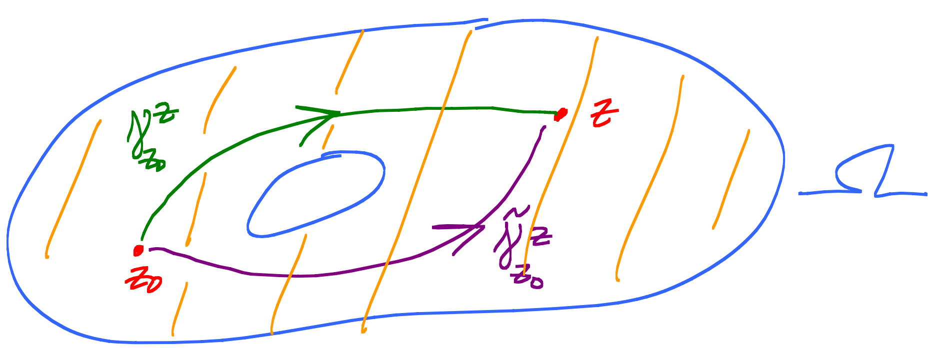
Note: Threw away constant term $-F(z_0)$

"Define"

$$F(z) = \int_{\gamma_{z_0}^{z}} f(w)\,dw$$

and show that $F'(z) = f(z)$.

Step 1: Show that F is well defined.



Aha! $$\int_{\gamma_{z_0}^{z}} - \int_{\tilde{\gamma}_{z_0}^{z}} = \int_{(\gamma_{z_0}^{z} \cup -\tilde{\gamma}_{z_0}^{z})}$$ closed curve

$= 0$ for f by assumption.

So def^n is <u>independent of path</u>. ✓

<u>Step 2</u>: Show $F' = f$